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Averaged Lagrangians and the mean dynamical effects of fluctuations in continuum mechanics

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Abstract

We begin by placing the Generalized Lagrangian Mean (GLM) equations for a compressible adiabatic fluid into the Euler-Poincaré (EP) variational framework of fluid dynamics, for an averaged Lagrangian. We then state the EP Averaging Lemma – that GLM averaged equations arise from GLM averaged Lagrangians in the EP framework. Next, we derive a set of approximate small amplitude GLM equations ($g\ell m$ equations) at second order in the fluctuating displacement of a Lagrangian trajectory from its mean position. These equations express the linear and nonlinear back-reaction effects on the Eulerian mean fluid quantities by the fluctuating displacements of the Lagrangian trajectories in terms of their Eulerian second moments.

The derivation of the $g\ell m$ equations uses the linearized relations between Eulerian and Lagrangian fluctuctions, in the tradition of Lagrangian stability analysis for fluids. The $g\ell m$ derivation also uses the method of averaged Lagrangians, in the tradition of wave, mean flow interaction (WMFI). The new $g\ell m$ EP motion equations for compressible and incompressible ideal fluids are compared with the Euler-alpha turbulence closure equations. An alpha model is a GLM (or $g\ell m$) fluid theory with a Taylor hypothesis closure (THC). Such closures are based on the linearized fluctuation relations that determine the dynamics of the Lagrangian statistical quantities in the Euler-alpha equations. Thus, by using the EP Averaging Lemma, we bridge between the GLM equations and the Euler-alpha closure equations, upon making the small-amplitude approximation resulting in the new $g\ell m$ equations in the EP variational framework. The $g\ell m$ equations also lead to generalizations of the Euler-alpha models to include compressibility and magnetic fields.

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1 Introduction

1.1 Decomposition of multiscale problems & scale-up

In turbulence, in climate modeling and in other multiscale fluids problems, a major challenge is "scale-up." This is the challenge of deriving models that correctly capture the mean, or large scale flow – including the influence on it of the rapid, or small scale dynamics.

In classical mechanics this sort of problem has been approached by choosing a proper "slow + fast" decomposition and deriving evolution equations for the slow mean quantities by using, say, the method of averages. See e.g., Sanders & Verhulst [1985], and Lochak & Meunier [1988] for descriptions of this method. For nondissipative systems in classical mechanics that arise from Hamilton's variational principle, the method of averages may extend to the averaged Lagrangian method, under certain conditions. See, e.g., Whitham [1965, 1970] for discussions of these conditions. In applying this method, the averaged Lagrangian acquires additional symmetries that are induced by the averaging process. These symmetries imply additional conservation laws for "adiabatic invariants" obtained via Noether's theorem. Thus, perhaps not unexpectedly, the system that results after averaging has fewer actively coupled degrees of freedom than in the original system, because averaging creates ignorable coordinates. This is an example of Lagrangian reduction by symmetries – the counterpart for Lagrangian systems of Marsden-Weinstein reduction for Hamiltonian systems, introduced in Marsden & Scheurle [1993].

Lagrangian reduction for mechanics on Lie groups. Some additional features arise in Lagrangian reduction when the system's state space is a Lie group, G, as in the case of fluid dynamics. In particular, suppose that a Lagrangian defined on the tangent space of a Lie group is also invariant under the group's action. Then a reduced Lagrangian may be defined on the group's Lie algebra (i.e., the tangent space at its identity). Such a reduction can be accomplished, for example, by transforming the Lagrangian to group invariant variables. The Euler-Lagrange equations on the group's cotangent space then reduce to **Euler-Poincaré (EP) equations** describing motion on the dual of its Lie algebra. This EP framework is now well developed. See Marsden & Ratiu [1999] for the an excellent introduction to the classical theory.

For continuum mechanics, see Holm, Marsden & Ratiu [1998a], who provide proofs of the basic EP theorems and present many applications of them for ideal fluids and plasmas with advected quantities.

In fluid dynamics, G is the group of diffeomorphisms – the smooth invertible maps that take the reference configuration of the fluid into its current configuration. Correspondingly, the Euler-Lagrange dynamics occurs in the Lagrangian picture of fluid motion, and the Euler-Poincaré dynamics occurs in the Eulerian picture. The invariance of the Lagrangian under the action of the group G arises in fluids whenever the variational principle depends only on Eulerian variables. These Eulerian variables are invariant under redefinitions of the Lagrangian coordinates by the action of the group (particle relabeling symmetry).

The EP theorems in the forms we shall use them for modeling in fluid dynamics are stated in Appendix #1 (section 8). There are two main EP theorems for continuum theories. The first EP theorem establishes that the two motion equations and also the two variational principles for the Lagrangian and Eulerian pictures of continua are *all* equivalent. The second EP theorem establishes the Kelvin-Noether circulation law as a result of Lagrangian reduction by particle relabeling symmetry. These two main EP theorems describe how the transport structure of Eulerian fluid dynamics arises variationally, after factoring the Lagrangian by its particle relabeling symmetry. For the continuum theory, see Holm, Marsden & Ratiu [1998a], and for its basis in classical mechanics, see Marsden & Ratiu [1999].

Eulerian vs Lagrangian means. In meteorology and oceanography, the averaging approach has a venerable history and many facets. Often this averaging is applied in the geosciences in combination with additional approximations involving force balances (for example, geostrophic and hydrostatic balances). It is also sometimes discussed as an initialization procedure that seeks a nearby invariant "slow manifold" Leith [1980]. Moreover, in meteorology and oceanography, the averaging may be performed in either the Eulerian, or the Lagrangian description. The relation between averaged quantities in the Eulerian and Lagrangian descriptions is one of the classical problems of fluid dynamics.

Wave, mean-flow interaction (WMFI). An example of a slow + fast decomposition appearing in meteorology and oceanography is the wave, mean-flow interaction (WMFI) problem. For reports of recent progress in the WMFI problem see, e.g., Grimshaw [1984] and Gjaja & Holm [1996]. In the WMFI problem, Lagrangian trajectories are represented as a mean-flow trajectory plus a WKB wave packet. The wave packet is taken as a small-amplitude rapid fluctuation with a slowly varying envelope. That is, the wave packet has slowly varying amplitude, frequency and wavenumber, but it has rapidly varying phase. The rapid phase sets the fast time scale over which one averages to obtain the equations for the wave, mean-flow interaction. Averaging in the Lagrangian over the fast phase of the wave in the WMF decomposition introduces a phase symmetry that leads via Noether's theorem to conservation of the adiabatic invariant called wave action density. The WMF approach for waves in fluids has its counterpart in the eikonal approximation for the physics of waves propagating in slowly varying media. As in the eikonal approximation, averaging over rapid phases in the WMFI problem leads to modulation equations. These WMFI modulation equations describe the fluid interacting with wave packets in the WKB geometrical ray optics approximation at leading order, with small corrections for dispersion.

The standard WMFI equations are a special case of the GLM equations of Andrews & McIntyre [1978a,b] in which the fluctuating displacement of the Lagrangian trajectory is given as a WKB wave packet. The GLM equations give an exact theory of nonlinear waves on a Lagrangian-mean flow for an arbitrary, but prescribed, representation of the fluctuations. See Gjaja & Holm [1996] for a detailed description of the various levels of approximation in the WMFI theory in terms of their variational principles. This reference also discusses the relation of the WMFI theory to the GLM equations, expressed in terms of the Lagrangian-mean of Hamilton's principle for incompressible rotating stratified fluids. For convenience, these levels of approximation in WMFI theory are also listed briefly in Appendix #3 (section 10).

Generalized Lagrangian mean (GLM). The GLM equations of Andrews & McIntyre [1978a] systematize the approach to Lagrangian fluid modeling by introducing a slow + fast decomposition of the Lagrangian particle trajectory in general form, then relating the Lagrangian mean

of a fluid quantity at the mean particle position to its Eulerian mean, evaluated at the displaced fluctuating position. The GLM equations are expressed directly in the Eulerian representation. The Lagrangian mean has the advantage of preserving the fundamental transport structure of fluid dynamics. For example, the Lagrangian mean commutes with the scalar advection operator and it preserves the Kelvin circulation property of the fluid motion equation.

Compatibility of averaging and reduction of Lagrangians. In making slow + fast decompositions and constructing averaged Lagrangians for fluid dynamics, care must generally be taken to see that the averaging and reduction procedures do not interfere with each other. (Reduction in this context refers to symmetry reduction of the action principle by the subgroup of the diffeomorphisms that takes the Lagrangian representation to the Eulerian representation of the flow field.)

This compatibility requirement is handled automatically in the GLM approach. The Lagrangian mean of the action principle for fluids does not interfere with its reduction to the Eulerian representation, since the averaging process is performed at *fixed Lagrangian coordinate*. After the Lagrangian mean is taken, the remaining particle relabeling symmetry in the action principle may be modded-out by reducing with respect to *another* diffeomorphism subgroup; namely, the subgroup that leaves invariant the quantities in the Eulerian specification of the *averaged* fluid flow. Thus, the process of taking the Lagrangian mean is compatible with reduction by the particle-relabeling group.

We shall perform this reduction of the action principle and thereby place the GLM equations into the EP framework. In doing this, we shall demonstrate the **variational reduction property of the Lagrangian mean**. This is encapsulated in the

> **EP Averaging Lemma.** *GLM averaging preserves the Euler-Poincaré (EP) variational framework that implies the GLM fluid equations.*

Thus, the Lagrangian mean's preservation of the fundamental transport structure of fluid dynamics also extends to preserving the EP variational structure of these equations. Of course, this extension is not possible with the Eulerian mean, because the Eulerian mean does *not* preserve the transport structure of Eulerian fluid mechanics.

Approach and main results. This paper begins by placing the exact nonlinear GLM equations for a compressible adiabatic fluid into the Euler-Poincaré (EP) variational framework of fluid dynamics for the corresponding GLM averaged Lagrangian. This result demonstrates the EP Averaging Lemma, that GLM averaged equations follow from GLM averaged Lagrangians in the EP framework. Thus, the EP Averaging Lemma puts the GLM averaged-Lagrangian approach and the method of GLM-averaged equations onto equal footing.[1] We also compare the GLM result with WMFI equations in the EP framework.

Staying in the EP framework, we shall derive a set of approximate small amplitude GLM equations ($g\ell m$ equations) by expanding the GLM Hamilton's principle to second order in the fluctuating displacement of a Lagrangian trajectory from its mean position. The resulting $g\ell m$ equations possess linear and nonlinear terms that express back-reaction effects on the mean motion due to the fluctuating displacements of the Lagrangian trajectories. The coefficients in these back-reaction terms involve *Eulerian mean* second moments of the fluctuating Lagrangian displacements. These back-reaction terms appear both in the mean stress tensor and in the total mean momentum. Thus, these terms have both linear and nonlinear effects.

The EP derivation of the $g\ell m$ equations uses the linearized relations between Eulerian and Lagrangian fluctuctions, in the tradition of Lagrangian stability analysis for fluids. The $g\ell m$ derivation also uses the method of averaged Lagrangians, in the tradition of wave, mean flow interaction (WMFI). We compare the new $g\ell m$ EP motion equations for compressible and incompressible ideal fluids with the Euler-alpha turbulence closure equations. We discuss closure hypotheses based on the linearized fluctuation relations that relate the $g\ell m$ equations to the Euler-alpha equations.

Thus, the new $g\ell m$ equations result as a bridge between the GLM equations and the Euler-alpha closure equations. This bridge is constructed by making a small-amplitude approximation of the GLM equations in the EP variational framework. In this framework, we also derive another variant of the Euler-alpha model that includes the effects of **compressibility**, or **magnetic fields**.

[1] This is quite a bonus for *both* approaches to modeling fluids. The averaged-Lagrangian theory produces dynamics that can be verified directly by averaging the original equations, and the GLM-averaged equations inherit the conservation laws that are available from the symmetries of the Lagrangian.

Compressible $\overline{g\ell m}$ closure equations. The equations we propose for the compressible $\overline{g\ell m}$ closure model for the Eulerian mean fluid velocity $\bar{\mathbf{u}}$ consist of the EP motion equation and two auxiliary equations. The EP motion equation is given by

$$\boxed{\begin{aligned}(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\big(\bar{\mathbf{u}} - \bar{\mathbf{p}}\big) + \big(\bar{u}_k - \bar{p}_k\big)\nabla \bar{u}^k \;-\; \nabla\,\bar{\Pi}^{g\ell m} \\ +\, \tfrac{1}{\bar{D}}\Gamma_{k\,l}\nabla\overline{\xi^k\xi^l} \;=\; \tfrac{1}{\bar{D}}\big(\,\nabla\cdot\nu\,(\bar{D} - \hat{\mathcal{O}})\nabla\big)\bar{\mathbf{u}}\,.\end{aligned}}$$

Here the pseudomomentum density $\bar{D}\bar{\mathbf{p}}$ and the symmetric operator $\hat{\mathcal{O}}$ are defined by

$$\bar{D}\bar{\mathbf{p}} = \hat{\mathcal{O}}\bar{\mathbf{u}} = \tfrac{1}{2}\left[\hat{\Delta}(\bar{D}\bar{\mathbf{u}}) + \bar{D}\hat{\Delta}\bar{\mathbf{u}} + \bar{\mathbf{u}}\,\hat{\Delta}\bar{D}\right],$$

where $\hat{\Delta} = \partial_l\,\overline{\xi^k\xi^l}\,\partial_k$ is a generalized Laplacian operator.

The Eulerian mean density $\bar{D}$ and statistical moments $\overline{\xi^k\xi^l}$ of the Lagrangian fluctuating particle displacements satisfy two auxiliary equations,

$$\partial_t\,\bar{D} + \mathrm{div}\bar{D}\bar{\mathbf{u}} = 0 \quad\text{and}\quad (\partial_t + \bar{\mathbf{u}}\cdot\nabla)\,\overline{\xi^k\xi^l} = 0\,.$$

The boundary conditions in the absence of kinematic shear viscosity ν are

$$\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0 \quad\text{and}\quad \hat{\mathbf{n}}\cdot\overline{\xi\xi} = 0 \quad\text{on the boundary.}$$

Here $\xi(\mathbf{x},t)$ is the fluctuating displacement of a Lagrangian trajectory away from its mean position, $\mathbf{x}$. These boundary conditions require the mean velocity and the Lagrangian fluctuating displacement both to be tangential at the boundary. (An additional boundary condition is needed when kinematic shear viscosity is present.)

Lagrangian averaged Navier-Stokes$-\alpha$ (LANS$-\alpha$) models. The barotropic *compressible* LANS$-\alpha$ model results from these $\overline{g\ell m}$ equations when the preserved initial condition $\overline{\xi^k\xi^l} = \alpha^2\delta^{kl}$ is chosen and the **Lagrangian correlation length scale** α is taken to be a constant, by choosing it to be so initially. For a constant α, the generalized Laplacian $\hat{\Delta}$ in the operator $\hat{\mathcal{O}}$ reduces to $\hat{\Delta} \to \alpha^2\Delta$, where Δ is the ordinary Laplacian.

The *incompressible* LANS$-\alpha$ model results when, in addition, the relation $\bar{D}=1$ is also enforced in these equations through the pressure constraint. In this case, the operator $\bar{D}-\hat{\mathcal{O}}$ reduces to the Helmholtz operator, $1-\alpha^2\Delta$. The incompressible LANS$-\alpha$ model using this operator has recently been proposed and tested as a **turbulence closure model** in Chen et al. [1998], [1999a,b,c]. See Foias, Holm & Titi [2001a,b] for reviews of the mathematical and physical properties of the incompressible LANS$-\alpha$ model. See also Marsden and Shkoller [2001] and references therein for additional analysis and discussions, as well as alternative derivations and interpretations of this model.

Outline of the paper. Section 2 introduces the information from the Generalized Lagrangian Mean (GLM) theory that we will need in the remainder of the paper. Section 3 begins by placing the GLM equations for a rotating adiabatic compressible fluid into the Euler-Poincaré (EP) variational framework of fluid dynamics in the Eulerian description. This is first done explicitly by re-deriving the GLM equations using Hamilton's principle with a GLM averaged Lagrangian. We then explain that every ideal GLM continuum equation follows from a GLM averaged variational principle, via the EP Averaging Lemma.

After re-framing GLM theory as an EP variational problem, we identify the parallels and similarities between the GLM and WMFI theories in the EP framework. We pause to correct an omission in the Andrews & McIntyre [1978a] GLM theory of stratified Boussinesq fluids that restores the Kelvin circulation theorem for these equations and its implication of local conservation for potential vorticity. We then move on to develop and discuss the $g\ell m$ small-amplitude approximation of GLM that also possesses this fundamental property.

In section 4, we review the standard linearized Eulerian/Lagrangian fluctuation relations. This is done in preparation for section 5 in which we construct a small-amplitude approximation of the GLM equations for a compressible adiabatic fluid at second order in the fluctuating displacement of a Lagrangian trajectory from its mean position. Substituting the *linear* fluctuation relations into the GLM action principle in the EP framework turns out to have both linear and nonlinear effects on the resulting EP equations. Another characteristic feature of these small amplitude GLM equations ($g\ell m$ equations) is that they involve second-gradients of Eulerian mean flow quantities, in combina-

tion with quadratic moments of the Lagrangian displacement statistics. The latter must be modeled in closing the system.

In section 6 we introduce several additional modeling decisions to arrive at second-order closures for the $g\ell m$ equations. These modeling decisions are formulated as variants of the Taylor-hypothesis for frozen-in turbulence. The resulting Taylor-hypothesis closure (THC) models recover the recently discovered Euler-alpha equations for incompressible ideal fluids. The THC models also provide a systematic basis for extending the Euler-alpha equations to the **compressible case**. Thus, the Euler-alpha equations reappear in a more general context than in their original derivation in Holm, Marsden & Ratiu [1998a,b]. Section 7 summarizes these Taylor-hypothesis closure results, provides additional discussion and makes suggestions for diagnosing direct numerical simulations for the purpose of determining the accuracy of these various approximate models.

Four appendices collect additional related material. Appendix #1 (section 8) states the two main EP theorems of Holm, Marsden & Ratiu [1998a] for fluids with advected properties. Appendix #2 (section 9) collects the linearized relations between the Eulerian and Lagrangian fluctuating fluid quantities and expresses them in a convenient geometrical (Lie-algebraic) form. Appendix #3 (section 10) lists the Lagrangian-mean Hamilton's principles for several levels of approximation in WMFI theory. (The GLM theory appears on this list of WMFI approximations at a certain level, because the *self-consistent* WMFI theory is more general than the GLM theory with *prescribed* fluctuation properties.) Appendix #4 (section 11) highlights some of the results in the rest of the paper by setting up the corresponding results for the GLM, $g\ell m$, and $\alpha-$models of ideal MHD (MagnetoHydroDynamics).

1.2 Eulerian and Lagrangian means

In fluids, averages over the fast variables in a slow + fast decomposition may be taken in either an Eulerian sense, or in a Lagrangian sense. The traditional Reynolds turbulence decomposition, for example, is taken in an Eulerian sense. In the Reynolds decomposition, the fluid velocity $\mathbf{u}$ at a given position, $\mathbf{x}$, is decomposed as $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ where $\overline{\mathbf{u}'} = 0$ and overbar (as in $\bar{\mathbf{u}}$) denotes the Eulerian mean. The Eulerian mean commutes with space and time partial derivatives, but

it has the **disadvantage** of not commuting with the advection operator $D/Dt = \partial_t + \mathbf{u}\cdot\nabla$, that appears thoughout the fluid equations. This disadvantage leads to the well-known closure problem – that the Eulerian mean fluid equations in general do not form a closed system. For example, the Eulerian mean equation for advection of a scalar fluid quantity χ gives

$$\overline{\Big(\frac{D\chi}{Dt}\Big)} = \frac{D^E\bar{\chi}}{Dt} \;+\; \underbrace{\overline{\mathbf{u}'\cdot\nabla\chi'}}_{\text{Extra}}, \quad \text{with} \quad \frac{D^E}{Dt} = \frac{\partial}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\,,$$

in which the extra term must be modeled to obtain closure.

The Lagrangian mean – denoted as $\bar{\chi}^L(\mathbf{x},t)$ for a fluid quantity χ – is taken at constant Lagrangian coordinate, $\mathbf{x}_0$. Consequently, by its definition the Lagrangian mean commutes with the advective derivative, written in the Lagrangian picture as

$$\overline{\big(\partial_t\big|_{\mathbf{x}_0}\chi\big)}^L = \partial_t\big|_{\mathbf{x}_0}\bar{\chi}^L\,.$$

Written in the Eulerian picture, this is

$$\overline{\Big(\frac{D\chi}{Dt}\Big)}^L = \frac{\partial\bar{\chi}^L}{\partial t} + \bar{\mathbf{u}}^L\cdot\nabla\bar{\chi}^L \equiv \frac{D^L\bar{\chi}^L}{Dt}\,,$$

where $\bar{\mathbf{u}}^L$ is the Lagrangian mean velocity, i.e., $\bar{\mathbf{u}}^L = D^L\mathbf{x}/Dt$ is the tangent vector along the mean Lagrangian trajectory.

Defining relation for Lagrangian mean. The Lagrangian mean for a fluid quantity χ is defined in terms of the Eulerian mean operation as

$$\bar{\chi}^L(\mathbf{x},t) = \overline{\chi\big(\mathbf{x}+\xi(\mathbf{x},t),t\big)}\,,$$

where $\overline{(\,\cdot\,)}$ denotes the Eulerian mean and $\mathbf{x}+\xi(\mathbf{x},t)$ is the current position of a fluid parcel whose mean position is $\mathbf{x}$. See Andrews & McIntyre [1978a,b] for discussions of the many implications of this defining relation. For example, one sees immediately from this defining relation that, relative to the Eulerian mean, the Lagrangian mean has **two disadvantages**: it is history dependent and it does not commute with the spatial gradient.

Thus, the Eulerian mean commutes with gradients, but it interferes with the advection operator and fails to produce a closed system of equations. In contrast, the Lagrangian mean commutes with the advection operator and produces the GLM equations. The GLM equations, however, are also not closed. For closure, they require the statistical properties of the Lagrangian disturbance ξ to be prescribed.

Levels of approximation. In principle, the GLM theory is more accurate than an Eulerian-mean theory, because it exactly preserves scalar advection relations and, thus, yields an exact nonlinear theory. As we shall see in section 3, the GLM theory also preserves the EP mathematical structure of the original unapproximated equations – including the Kelvin circulation theorem and the balance laws for energy and momentum. However, the results of any Lagrangian mean theory may be difficult to interpret accurately in an Eulerian setting.

For its completion, the GLM theory still requires **closure assumptions** about the Lagrangian-mean statistics of its wave properties – the Lagrangian fluctuating displacements, ξ. For sufficiently small disturbances, plausible assumptions based on WMFI theory may assist in this closure. However, at finite disturbance amplitude, the nonlinear effects of the mean motion on the statistics of the Lagrangian fluctuating displacements must become part of a *dynamically self-consistent* solution, and these statistics can no longer be taken as prescribed quantities. See Gjaja and Holm [1996] for initial investigations of a dynamically self-consistent WMFI theory at the level of the WKB envelope approximation.

Therefore, we see the need for developing an approximate small amplitude counterpart to the GLM theory, in preparation for developing a set of dynamically self-consistent closure assumptions. For this, we shall begin in section 3 by placing the the GLM equations themselves into the larger framework of EP variational principles.

In section 4 we shall recall the linearized Eulerian/Lagrangian fluctuation relations that are familiar from traditional Lagrangian stability analysis for fluids. In section 5, we shall apply these linearized fluctuation relations within the EP framework to develop a **small-amplitude approximation** that minimizes the disadvantages of interpreting the GLM description in an Eulerian setting. Namely, we shall assume that the fluctuating displacement of the fluid trajectory around its mean,

$\xi(\mathbf{x}, t)$, is sufficiently small and also that the mean flow is sufficiently smooth, that we may linearize the relations for Eulerian fluctuating quantities in terms of the Lagrangian displacement fluctuation, ξ. This is standard practice in traditional Lagrangian stability theory. These standard linearized relations will allow us to identify which moments of the Lagrangian statistics must be modeled in making an approximate **second-order Eulerian-mean closure** of the GLM equations of motion at order $O(|\xi|^2)$. These second-order small-amplitude GLM equations (denoted $g\ell m$ equations) are derived by applying the EP framework to the Eulerian mean of the second-variation Lagrangian for GLM.

In section 6, we shall seek **variational closures** of the $g\ell m$ equations. The closed $\overline{g\ell m}$ equations result, upon introducing an additional **Taylor hypothesis** into the derivation of $g\ell m$ theory. The $g\ell m$ theory is derived in section 5 by making approximations, averaging and taking variations of the GLM Lagrangian. The family of closed $\overline{g\ell m}$ equations will be derived in the same EP framework. However, we shall first introduce particular (or partial) solutions of the linearized fluctuation relations into the derivation of $g\ell m$ theory, before averaging and taking variations in the EP variational principle. The closed family of $\overline{g\ell m}$ equations includes the **Euler-alpha model** for incompressible ideal fluids discovered in Holm, Marsden & Ratiu [1998a,b], as well as recent extensions of it derived in Marsden & Shkoller [2001].

Finally, the $\overline{g\ell m}$ theory given here also extends these Euler-alpha equations to include the effects of **compressibility** and **magnetic fields**.

2 Brief review of GLM theory for compressible fluids

An exceptional accomplishment in formulating averaged motion equations for fluid dynamics is the Generalized Lagrangian Mean (GLM) theory of nonlinear waves on a Lagrangian-mean flow, as explained in two consecutive papers of Andrews & McIntyre [1978a,b]. This section introduces what we shall need later from the rather complete description given in these papers. (Even now, these fundamental papers still make worthwhile reading and are taught in many atmospheric science departments.) In section 3 we shall place the GLM equations into the

Euler-Poincaré variational framework. Then in section 5, we shall derive the second-order approximate $g\ell m$ equations in this variational framework by making a small-amplitude approximation.

2.1 Relevant information from the GLM theory

Defining relations for Lagrangian mean & Stokes correction in terms of Eulerian mean

The GLM equations are based on defining fluid quantities at a displaced fluctuating position. In the GLM description, $\bar\chi$ denotes the Eulerian mean of a fluid quantity $\chi = \bar\chi + \chi'$ while $\bar\chi^L$ denotes the Lagrangian mean of the same quantity, defined by

$$\bar\chi^L(\mathbf{x}) \equiv \overline{\chi^\xi(\mathbf{x})}\,, \quad \text{with} \quad \chi^\xi(\mathbf{x}) \equiv \chi(\mathbf{x} + \xi(\mathbf{x},t))\,.$$

Here $\mathbf{x}^\xi \equiv \mathbf{x} + \xi(\mathbf{x},t)$ is the current position of a Lagrangian fluid trajectory whose mean position is $\mathbf{x}$. Thus, $\xi(\mathbf{x},t)$ with vanishing Eulerian mean $\bar\xi = 0$ denotes the fluctuating displacement of a Lagrangian particle trajectory about its mean position $\mathbf{x}$.

> **Remark.** Fortunately, this notation is also *standard* in the stability analysis of fluid equilibria in the Lagrangian picture. See, e.g., the classic works of Bernstein et al. [1958], Frieman & Rotenberg [1960] and Newcomb [1962]. See Jeffrey & Taniuti [1966] for a collection of reprints showing applications of this approach in controlled thermonuclear fusion research. For insightful reviews, see Bernstein [1983], Chandrasekhar [1987] and, more recently, Hameiri [1998]. Rather than causing confusion, this confluence of notation encourages the transfer of ideas between traditional Lagrangian stability analysis for fluids and GLM theory, as we shall see in section 5.

In GLM theory, the difference $\chi^\xi - \bar\chi^L = \chi^\ell$ is called the **Lagrangian disturbance** of the quantity χ. One finds $\overline{\chi^\ell} = 0$, since the Eulerian mean possesses the **projection property** $\bar{\bar\chi} = \bar\chi$ for any quantity χ (and, in particular, it possesses that property for χ^ξ).[2] Andrews & McIntyre [1978a] show that, provided the smooth map

[2] Note that spatial filtering in general does *not* possess the projection property.

$\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x}, t)$ is invertible (that is, provided the vector field $\xi(\mathbf{x}, t)$ generates a diffeomorphism), then the Lagrangian disturbance velocity $\mathbf{u}^\ell$ may be expressed in terms of ξ by

$$\mathbf{u}^\ell = \mathbf{u}^\xi - \bar{\mathbf{u}}^L = \frac{D^L\xi}{Dt}\,, \quad \text{where} \quad \frac{D^L\xi}{Dt} \equiv \frac{\partial \xi}{\partial t} + \bar{\mathbf{u}}^L \cdot \nabla \xi\,.$$

Consequently, the Lagrangian disturbance velocity $\mathbf{u}^\ell$ is a genuine fluctuation quantity satisfying $\overline{\mathbf{u}^\ell} = 0$, since $\overline{\mathbf{u}^\xi} - \overline{\bar{\mathbf{u}}^L} = \overline{\mathbf{u}^\xi} - \overline{\overline{\mathbf{u}^\xi}} = 0$, by the projection property. (Alternatively, $\overline{\mathbf{u}^\ell} = \overline{D^L\xi/Dt} = 0$ also follows, since the Eulerian mean commutes with D^L/Dt and ξ has mean zero.)

The difference between the Eulerian and Lagrangian means is called the **Stokes correction**, e.g.,

$$\bar{\chi}^S(\mathbf{x}) = \bar{\chi}^L(\mathbf{x}) - \bar{\chi}(\mathbf{x})\,.$$

In a Taylor series approximation, one finds

$$\bar{\chi}^S = \overline{\xi \cdot \nabla \chi'} + \tfrac{1}{2}\overline{\xi\xi} : \nabla\nabla\bar{\chi} + O(|\xi|^3)\,.$$

The order $O(\overline{\xi\xi})$ terms in $\bar{\chi}^S$ may be neglected, provided the second gradients of the mean $\nabla\nabla\bar{\chi}$ are sufficiently small, as we shall assume henceforth.

2.2 GLM preserves advective transport relations, modulo their tensor transformation factors

GLM scalar advection relations

At position $\mathbf{x}$ the velocity $\mathbf{u}^\xi = \bar{\mathbf{u}}^L + \mathbf{u}^\ell$ is the sum of the Lagrangian mean velocity $\bar{\mathbf{u}}^L$ and the Lagrangian disturbance velocity $\mathbf{u}^\ell$. Thus, $\mathbf{u}^\xi = D^L\mathbf{x}^\xi/Dt$ and for any scalar field $\chi(\mathbf{x}, t)$ one has,

$$\Big(\frac{D\chi}{Dt}\Big)^\xi = \frac{D^L}{Dt}(\chi^\xi)\,.$$

Because $\bar{\mathbf{u}}^L$ appearing in the advection operator $D^L/Dt = \partial_t + \bar{\mathbf{u}}^L \cdot \nabla$ is a mean quantity, one then finds, as expected, that the Lagrangian mean $\overline{(\,\cdot\,)}^L$ commutes with the material derivative D/Dt. That is,

$$\overline{\Big(\frac{D\chi}{Dt}\Big)}^L = \frac{D^L}{Dt}(\bar{\chi}^L)\,, \quad \text{and} \quad \Big(\frac{D\chi}{Dt}\Big)^\ell = \frac{D^L}{Dt}\chi^\ell\,,$$

where $\chi^\ell = \chi^\xi - \bar{\chi}^L$ is the Lagrangian disturbance of χ satisfying $\overline{\chi^\ell} = 0$. For example, in an adiabatic compressible flow, the specific entropy s is advected as a scalar. That is, it satisfies $Ds/Dt = 0$ and, consequently, $D^L\bar{s}^L/Dt = 0$, as well. Hence, $s^\xi = \bar{s}^L$ follows, by integration of $D^L(\bar{s}^L - s^\xi)/Dt = 0$ along mean trajectories and invertibility of the map $\mathbf{x} \to \mathbf{x} + \xi(\mathbf{x}, t)$.

> **Remark.** Of course, this identification is also obvious physically, since the Lagrangian mean $\bar{s}^L$ and the current value s^ξ refer to the *same* Lagrangian fluid label. That is, we initialize with $\xi(\mathbf{x}_0, 0) = 0$, for a Lagrangian coordinate $\mathbf{x}_0 = \mathbf{x}(\mathbf{x}_0, 0)$. (This initialization still allows for random initial velocities.)

Mass conservation: the GLM continuity equation

Remarkably, $\bar{D}^L$ is not the density advected in the GLM theory. That is,

$$\partial_t\, \bar{D}^L + \mathrm{div}\bar{D}^L\bar{\mathbf{u}}^L \neq 0\,.$$

Instead, GLM satisfies another density advection relation – the **GLM continuity equation**,

$$\partial_t\, \tilde{D} + \mathrm{div}\tilde{D}\bar{\mathbf{u}}^L = 0\,, \tag{2.1}$$

for a density $\tilde{D}$, which is also a mean quantity. That is, $\bar{\tilde{D}} = \tilde{D}$, where we invoke again the projection property of the Eulerian mean. The GLM conserved density $\tilde{D}$ is given by

$$\boxed{\tilde{D} = D^\xi \mathcal{J}\,, \quad \text{where} \quad \mathcal{J} = \det\big(\nabla_{\mathbf{x}}(\mathbf{x} + \xi)\big)\,. \qquad (2.2)}$$

The GLM continuity equation for the density $\tilde{D}$ may be shown by transforming the instantaneous mass conservation relation $D^\xi\, d^3x^\xi = D(x_0)d^3x_0$ into

$$D^\xi \mathcal{J} \equiv D^\xi(x)\det(\nabla_{\mathbf{x}}(\mathbf{x} + \xi)) = D(x_0)d^3x_0/d^3x \equiv \tilde{D}$$

and then using the defining relation (1.2) for the Lagrangian mean in terms of the Eulerian mean. In taking the Eulerian mean of this

relation, we keep in mind that $\mathbf{x}$ is an average quantity, so the right hand side is *already* an average quantity. Thus, $\tilde{D} = D^\xi \mathcal{J}$ satisfies $\bar{\tilde{D}} = \tilde{D}$, as claimed, and we note that $\tilde{D} \neq \bar{D}^L$, in general. The mean mass conservation relation $\tilde{D} d^3x = D(x_0) d^3x_0$ then implies the continuity equation for $\tilde{D}$,

$$\partial_t \tilde{D} + \mathrm{div} \tilde{D} \bar{\mathbf{u}}^L = 0 \,,$$

upon recalling that $\bar{\mathbf{u}}^L$ is the velocity tangent to the mean Lagrangian position $\mathbf{x}$.

Remarks.

- The advective transport result for the density $\tilde{D}$ is especially clear when expressed in Lie derivative form as,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big[\Big(D(x_0) d^3x_0\Big) \cdot g^{-1}(t)\Big] \\ &= \Big(\frac{\partial}{\partial t} + \pounds_{\bar{\mathbf{u}}^L}\Big)(\tilde{D} d^3x) = \Big(\partial_t \tilde{D} + \mathrm{div} \tilde{D} \bar{\mathbf{u}}^L\Big) d^3x \,, \end{aligned}$$

  where $\cdot g^{-1}(t)$ denotes the right action of the group G and $\pounds_{\bar{\mathbf{u}}^L}$ denotes the Lie derivative with respect to $\bar{\mathbf{u}}^L$ (in the notation explained in Appendices #1 and # 2, sections 8 and 9).

- For a fluid with **constant unit density**, $D^\xi = 1$, the GLM theory gives

$$\tilde{D} = \overline{D^\xi \mathcal{J}} = \overline{\det\big(\nabla_{\mathbf{x}}(\mathbf{x} + \xi)\big)} = 1 - \tfrac{1}{2}\big(\overline{\xi^k \xi^\ell}\big)_{,k\ell} + O(|\xi|^3) \,.$$

  Hence, for constant instaneous density, the Lagrangian mean velocity $\bar{\mathbf{u}}^L$ has an order $O(|\xi|^2)$ divergence,

$$\mathrm{div}\, \bar{\mathbf{u}}^L = -\frac{1}{\tilde{D}} \frac{D^L \tilde{D}}{Dt} = \frac{1}{2} \frac{D^L}{Dt} \big(\overline{\xi^k \xi^\ell}\big)_{,k\ell} + O(|\xi|^3) \,,$$

  as shown in Andrews & McIntyre [1978a].

Magnetic flux advection: the GLM frozen-in magnetic field

A preserved mean magnetic flux equation is provided as a basic mean advection relation in developing a GLM theory of ideal MHD.

The magnetic flux advection relation for a magnetic field in ideal MHD (MagnetoHydroDynamics) is

$$\partial_t \mathbf{B}^\xi = \big(\mathrm{curl}\,(\mathbf{u}\times\mathbf{B})\big)^\xi \quad \text{with} \quad (\mathrm{div}\,\mathbf{B})^\xi = 0\,.$$

The corresponding GLM advection law may be obtained by averaging the Cauchy solution for this relation

$$\frac{\mathbf{B}^\xi}{D^\xi}\cdot\frac{\partial x^i}{\partial \mathbf{x}^\xi} = \frac{\mathbf{B}(\mathbf{x}_0)}{D(\mathbf{x}_0)}\cdot\frac{\partial x^i}{\partial \mathbf{x}_0}\,,$$

in which the initial condition $\mathbf{B}(\mathbf{x}_0)$ has zero divergence. Upon taking the mean of this relation and using $\tilde{D} = \overline{D^\xi \mathcal{J}}$ for the mean GLM density, we find that the divergenceless vector $\tilde{\mathbf{B}}$ whose components are

$$\tilde{B}^i = K^i_j\, B^{\xi\, j} \quad \text{with} \quad K^i_j = \mathcal{J}\,\frac{\partial x^i}{\partial x^{\xi\, j}}\,, \quad i,j = 1,2,3\,,$$

is also a mean quantity. That is, $\bar{\tilde{\mathbf{B}}} = \tilde{\mathbf{B}}$ and we note that $\tilde{\mathbf{B}} \neq \bar{\mathbf{B}}^L$. Here K^i_j satisfying $K^i_j\,(\partial x^{\xi\, j}/\partial x^k) = \mathcal{J}\delta^i_k$ is the matrix K of cofactors of the Jacobian $\mathsf{J} \equiv \partial \mathbf{x}^\xi/\partial \mathbf{x}$. The mean vector quantity $\tilde{\mathbf{B}} = \mathsf{K}\cdot\mathbf{B}^\xi$ then satisfies the magnetic flux advection relation,

$$\partial_t\,\tilde{\mathbf{B}} = \mathrm{curl}\,(\bar{\mathbf{u}}^L\times\tilde{\mathbf{B}}) \quad \text{with} \quad \mathrm{div}\,\tilde{\mathbf{B}} = 0\,.$$

That is, the flux of the divergenceless mean quantity $\tilde{\mathbf{B}}$ is "frozen" into the Lagrangian mean motion of the fluid.

A fully three dimensional nonlinear GLM theory of MHD is still under development. See, however, Grappin, E. Cavillier & M. Velli [1997] for a recent account, based on the WKB approximation, of the propagation of acoustic waves incident on the base of a stellar wind and their back-reaction on the mean flow, in the spherically symmetric, isothermal case.

Advection of a covariant symmetric tensor

When nonlinear elasticity is also a factor in the fluid evolution, there is an additional advection relation,

$$\frac{\partial}{\partial t} S^{\xi}_{ab} = -\left(u^k S_{ab,k} + S_{kb} u^k_{,a} + S_{ka} u^k_{,b}\right)^{\xi} .$$

This is the advection relation for the Cauchy-Green strain tensor $S_{ab}(\mathbf{x}^{\xi}, t)$, which measures nonlinear strain in *Eulerian* coordinates. The corresponding Cauchy solution of this advection relation is expressed in a Cartesian tensor basis as

$$(S_{kl}\, dx^k \otimes dx^l)^{\xi} = S_{ab}(\mathbf{x}_0)\, dx_0^a \otimes dx_0^b \, .$$

The GLM advection law for such a covariant symmetric tensor may be obtained by rearranging this Cauchy solution. Hence, the symmetric tensor,

$$\tilde{\mathsf{S}}_{ij} \equiv (\mathsf{J}^T \cdot \mathsf{S}^{\xi} \cdot \mathsf{J})_{ij} = S^{\xi}_{kl} \frac{\partial x^{\xi\, k}}{\partial x^i} \frac{\partial x^{\xi\, l}}{\partial x^j} = S_{ab}(\mathbf{x}_0) \frac{\partial x_0^a}{\partial x^i} \frac{\partial x_0^b}{\partial x^j} \, ,$$

is a mean quantity, where the Jacobian matrix J has elements $\mathsf{J}^k_i \equiv \partial x^{\xi\, k}/\partial x^i$. The corresponding GLM advection law for the mean Cauchy-Green strain tensor $\tilde{\mathsf{S}}$ is given by

$$\frac{\partial}{\partial t} \tilde{S}_{ab} = -\, \bar{u}^{L\, k}\, \tilde{S}_{ab,k} - \tilde{S}_{kb}\, \bar{u}^{L\, k}_{,a} - \tilde{S}_{ka}\, \bar{u}^{L\, k}_{,b} \, . \tag{2.3}$$

Note that $\tilde{\mathsf{S}} = \bar{\tilde{\mathsf{S}}}$ is a mean fluid quantity, but $\tilde{\mathsf{S}} \neq \bar{\mathsf{S}}^L$ because of the additional factors involving the Jacobian J.
Remark.

> These examples demonstrate that the GLM theory preserves the advective transport structure of fluid dynamics, modulo the means of their tensor transformation factors. On a Riemannian manifold, additional metric terms are also present in these tensor transformation factors. See, e.g., Marsden and Shkoller [2001] and references therein.

2.3 GLM motion equations for adiabatic compressible fluids

The GLM motion equation for adiabatic compressible fluids in a frame rotating with constant frequency Ω are given in Andrews & McIntyre [1978a] in Cartesian coordinates as

$$\boxed{\frac{D^L}{Dt}\big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}}\big) + \big(\bar{u}^L_k - \bar{p}_k\big)\nabla \bar{u}^L_k + 2\Omega \times \bar{\mathbf{u}}^L + \nabla\,\Pi - \bar{T}^L \nabla \bar{s}^L = 0\,. \quad (2.4)}$$

Here $\bar{\mathbf{p}}$ is the **pseudomomentum vector**, a mean quantity defined by,

$$\bar{\mathbf{p}} \equiv -\,\overline{[u^\ell_k + (\Omega \times \xi)_k]\nabla \xi^k}\,. \tag{2.5}$$

The mean potential Π has the form,

$$\Pi = \overline{h(p^\xi, s^\xi)} + \bar{\Phi}^L(\mathbf{x}) - \frac{1}{2}\overline{\mathbf{u}^\xi \cdot [\mathbf{u}^\xi + 2\Omega \times \xi]}\,, \tag{2.6}$$

in which

$$\overline{h(p^\xi, s^\xi)} \equiv \overline{e(D^\xi, s^\xi)} + \overline{(p^\xi / D^\xi)}$$

is the mean specific enthalpy and $\bar{\Phi}^L$ is the Lagrangian mean of an external potential Φ. We note that $s^\xi = \bar{s}^L$ since the specific entropy is a Lagrangian variable in the adiabatic case. The partial derivative $\bar{T}^L = \partial \overline{e(D^\xi, \bar{s}^L)} / \partial \bar{s}^L$ is the Lagrangian mean temperature.

For an adiabatic compressible fluid, the thermodynamic First Law following a fluid parcel is

$$de(D^\xi, \bar{s}^L) = -\,p^\xi\, d\Big(\frac{1}{D^\xi}\Big) + T^\xi d\bar{s}^L\,.$$

Hence, its Eulerian mean becomes, upon using $\tilde{D} = D^\xi \mathcal{J}$ from mass conservation,

$$\boxed{d\,\overline{e(D^\xi, \bar{s}^L)} = -\,\frac{1}{\tilde{D}}\,\overline{(p^\xi\, d\mathcal{J})} + \frac{1}{\tilde{D}}\,\overline{(p^\xi / D^\xi)}\, d\tilde{D} + \bar{T}^L d\bar{s}^L\,. \qquad (2.7)}$$

Remarks.

- The determinant $\mathcal{J} = \det\big(\nabla_{\mathbf{x}}(\mathbf{x}+\xi)\big)$ is a fluctuating quantity, not a mean fluid quantity. Therefore, $\mathcal{J}$ will not contribute to variations with respect to mean fluid quantities. However, $\delta\mathcal{J} = K^j_k(\partial\,\delta\xi^k/\partial x^j)$ with cofactor $K^j_k = \mathcal{J}\partial\, x^j/\partial(x^k+\xi^k)$ does contribute to variations with respect to ξ in the self-consistent WMFI theory of Gjaja & Holm [1996]. Such variations also arise, for example, in the Lagrangian stability analysis of the equilibrium solutions of the GLM equations. See, e.g., Andrews & McIntyre [1978b] for a discussion of Hamilton's principle for the Lagrangian disturbance ξ and its relation to the wave action density of the WMFI theory.

- For an elastic medium, the specific internal energy $e(D, b, S_{ab})$ also depends on the Cauchy-Green strain tensor S_{ab}. In this case, the stress tensor per unit mass σ^{ab} is determined from the equation of state by the Doyle-Erickson formula $\sigma^{ab} \equiv \partial e/\partial S_{ab}$. The Eulerian mean First Law relation for the specific internal energy of such an elastic medium then contains an additional stress term,

  $$d\overline{e^\xi} = -\,\overline{p^\xi d(1/D^\xi)} + \bar{T}^L d\bar{s}^L + \overline{(\sigma^\xi)^{ab} dS^\xi_{ab}}\,.$$

  The corresponding mean stresses will be defined from the average quantity

  $$\overline{(\sigma^\xi)^{ab} d\,(\mathsf{J}^{-T}\!\cdot\tilde{\mathsf{S}}\cdot\mathsf{J}^{-1})_{ab}} = \overline{(\sigma^\xi)^{ab}(\mathsf{J}^{-T}\!\cdot d\,\tilde{\mathsf{S}}\cdot\mathsf{J}^{-1})_{ab}} + \text{ terms in } \xi\,.$$

- Thus, the preservation of advective transport relations in the GLM theory enables proper definitions of the **thermodynamic derivatives of mean constitutive relations** with respect to GLM average fluid variables.

2.4 Pseudomomentum & the transport structure of the GLM motion equation

The significance of the pseudomomentum $\mathbf{p}$ to the transport structure of the GLM equations can be understood from the Lagrangian mean of the contour integral appearing in Kelvin's circulation theorem for fluid motion in a rotating frame. The rotation frequency Ω is allowed to depend on position and is given by $2\Omega(\mathbf{x}^\xi) = (\operatorname{curl}\mathbf{R})^\xi$. The rotation potential $\mathbf{R}(\mathbf{x}^\xi)$ is decomposed in standard GLM fashion as $\mathbf{R}^\xi = \bar{\mathbf{R}}^L +$

$\mathbf{R}^\ell$. (A constant rotation frequency is recovered from specializing to $\mathbf{R}^\xi = \Omega \times \mathbf{x}^\xi$.)

The GLM average of Kelvin's circulation integral is defined as,

$$
\begin{aligned}
\overline{I(t)} &= \overline{\oint_{\gamma^\xi(t)} \big(\mathbf{u}^\xi + \mathbf{R}(\mathbf{x}^\xi)\big)\cdot d\mathbf{x}^\xi} \\
&= \overline{\oint_{\gamma^\xi(t)} \big(\bar{\mathbf{u}}^L + \bar{\mathbf{R}}^L + \mathbf{u}^\ell + \mathbf{R}^\ell\big)\cdot (d\mathbf{x} + d\xi)} \\
&= \oint_{\bar{\gamma}^L(t)} \big(\bar{\mathbf{u}}^L + \bar{\mathbf{R}}^L + \overline{[u^\ell_k + R^\ell_k]\nabla\xi^k}\ \big)\cdot d\mathbf{x} \\
&= \oint_{\bar{\gamma}^L(t)} (\bar{\mathbf{u}}^L + \bar{\mathbf{R}}^L - \mathbf{p})\cdot d\mathbf{x}\,,
\end{aligned}
$$

where the contour $\bar{\gamma}^L(t)$ moves with velocity $\bar{\mathbf{u}}^L$, since it follows the fluid parcels as the average is taken. Thus, the Lagrangian mean leaves invariant the *form* of the Kelvin integral, while averaging the *velocity* of its contour. In addition, the pseudomomentum vector $\mathbf{p}$ defined in (2.5) appears in the GLM averaged Kelvin integral as the Lagrangian mean contribution of the fluctuations to the GLM averaged *integrand.*

The time derivative of the GLM averaged Kelvin circulation integral is,

$$
\frac{d}{dt}\overline{I(t)} = \oint_{\bar{\gamma}^L(t)} \Big[(\partial_t + \bar{\mathbf{u}}^L\cdot\nabla)(\bar{\mathbf{u}}^L - \mathbf{p}) + (\bar{u}^L_k - p_k)\nabla\bar{u}^{L\,k} + 2\Omega\times\bar{\mathbf{u}}^L\Big]\cdot d\mathbf{x}\,.
$$

The combination of terms in the integrand defines the **transport structure** of the GLM theory. From the GLM motion equation (2.4) one now finds the GLM Kelvin circulation theorem for adiabatic compressible flow,

$$
\frac{d}{dt}\overline{I(t)} = \frac{d}{dt}\oint_{c(\bar{\mathbf{u}}^L)} \Big(\bar{\mathbf{u}}^L + \bar{\mathbf{R}}^L - \mathbf{p}\Big)\cdot d\mathbf{x} = \oint_{c(\bar{\mathbf{u}}^L)} \bar{T}^L\, d\bar{s}^L\,.
$$

Thus, the Lagrangian mean *averages the velocity* of the fluid parcels on the Kelvin circulation loop, while it *adds the mean contribution* of the fluctuations to the Kelvin circulation integrand. In particular, upon taking the Lagrangian mean, the velocity of fluid parcels on the circulation loop and the velocity appearing in the circulation integrand are *different.*

In the isentropic case (or, if the loop $c(\bar{\mathbf{u}}^L)$ moving with the Lagrangian mean flow lies entirely on a level surface of $\bar{s}^L$) then the right hand side vanishes, and one finds the "generalized Charney-Drazin theorem" for transient waves discussed in Andrews & McIntyre [1978a].

3 EP formulation of the GLM equations using an averaged variational principle

3.1 EP Averaging Lemma for GLM equations

Most of the important properties of the GLM equations are discussed in Andrews & McIntyre [1978a]. Many of these properties arise from general mathematical structures that are shared by all exact nonlinear ideal fluid theories. With the help of the thermodynamic identity (2.7) for $d\,\overline{e(D^\xi, \bar{s}^L)}$ we shall recast the GLM fluid motion equation (2.4) as an **Euler-Poincaré (EP) equation**,

$$\boxed{\frac{\partial}{\partial t}\frac{\delta\bar{\ell}}{\delta\bar{u}_i^L} + \frac{\partial}{\partial x_k}\Big(\frac{\delta\bar{\ell}}{\delta\bar{u}_i^L}\bar{u}_k^L\Big) + \frac{\delta\bar{\ell}}{\delta\bar{u}_k^L}\frac{\partial\bar{u}_k^L}{\partial x_i} = \tilde{D}\frac{\partial}{\partial x_i}\frac{\delta\bar{\ell}}{\delta\tilde{D}} - \frac{\delta\bar{\ell}}{\delta\bar{s}^L}\frac{\partial\bar{s}^L}{\partial x_i}\,, \qquad (3.1)}$$

expressed in terms of variational derivatives of an averaged Lagrangian, $\bar{\ell}(\bar{\mathbf{u}}^L, \tilde{D}, \bar{s}^L)$. See Holm, Marsden & Ratiu [1998a,b] for an exposition of the mathematical structures that arise in the EP theory of ideal fluids that possess advected quantities such as heat and mass. For GLM the Eulerian expression of the averaged Lagrangian is

$$\boxed{\begin{aligned}\bar{\ell}(\bar{\mathbf{u}}^L, \tilde{D}, \bar{s}^L) \;=\; \int d^3x\,\tilde{D}\Big[\frac{1}{2}\overline{\Big|\bar{\mathbf{u}}^L + \frac{D^L\xi}{Dt}\Big|^2} + \overline{(\Omega\times\mathbf{x}^\xi)\cdot\Big(\bar{\mathbf{u}}^L + \frac{D^L\xi}{Dt}\Big)} \\ - \overline{\Phi(\mathbf{x}^\xi)} - \overline{e(D^\xi, \bar{s}^L)}\Big]\,. \qquad (3.2)\end{aligned}}$$

The mean Lagrangian $\bar{\ell} \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}^L, \tilde{D}, \bar{s}^L; \xi)d^3x$ is a straight transcription of the standard Lagrangian for adiabatic fluids into the GLM formalism, followed by taking the Eulerian mean. If desired, the rotation frequency can be allowed to depend on position by replacing $(\Omega\times\mathbf{x}^\xi) \to \mathbf{R}(\mathbf{x}^\xi)$, in which case $2\Omega \to (\mathrm{curl}\,\mathbf{R})^\xi$. The variational

derivatives of $\bar{\ell}$ are given by

$$\delta\bar{\ell} \;=\; \int d^3x\,\Big[\tilde{D}\Big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}} + \Omega\times\mathbf{x}\Big)\cdot\delta\bar{\mathbf{u}}^L - \;\tilde{D}\,\bar{T}^L\,\delta\bar{s}^L - \Pi\,\delta\tilde{D} \qquad (3.3)$$
$$+\,\tilde{D}\,\overline{[\,u^\ell_k + (\Omega\times\xi)_k](\partial_t\,\delta\xi^k + \bar{\mathbf{u}}^L\cdot\nabla\delta\xi^k)} + \overline{p^\xi K^j_k(\partial\,\delta\xi^k/\partial x^j)}\,\Big]\,,$$

where $K^j_k = J\partial\, x^j/\partial(x^k+\xi^k)$ is the cofactor that arises from the thermodynamic identity (2.7). Thus, the pseudomomentum $\bar{\mathbf{p}}$ defined in (2.5), the Lagrangian-mean temperature $\bar{T}^L = \partial\overline{e(D^\xi,\bar{s}^L)}/\partial\bar{s}^L$ and the potential Π in (2.6) all arise naturally in the variational derivatives of the Lagrangian $\bar{\ell}$ in (3.2) with respect to the mean fluid quantities.

One may verify that the GLM motion equation (2.4) for the mean fluid motion is now recovered by substituting the variational derivatives of $\bar{\ell}$ in $\bar{\mathbf{u}}^L$, $\tilde{D}$ and $\bar{s}^L$ into the EP equation (3.1). This computation places the GLM theory into the EP framework for the averaged Lagrangian (3.2) and, thus, directly proves the following.

Lemma 1 GLM adiabatic fluids satisfy EP equations *The GLM motion equation (2.4) for a compressible adiabatic fluid results when the Euler-Poincaré equation (3.1) is applied to the averaged Lagrangian (3.2).*

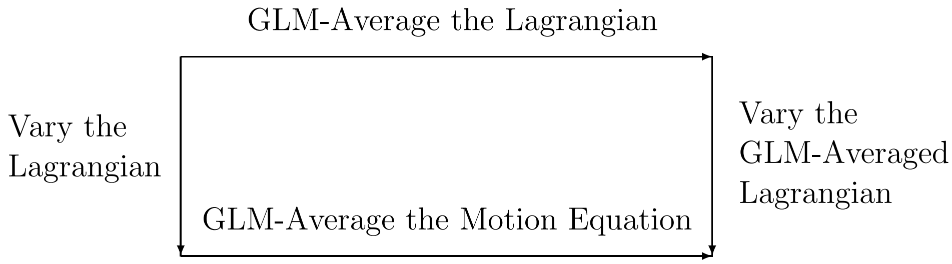
This result suggests that a much broader principle is operating, namely,

Lemma 2 EP Averaging Lemma *GLM-averaging preserves the four equivalence relations of the EP theorem.*

Andrews & McIntyre's exposition of the GLM theory was published in 1978, while the EP theory for advective fluid dynamics was only recently developed in Holm, Marsden and Ratiu [1998a]. Therefore, it may be more natural to state the EP Averaging Lemma from the viewpoint of a variational principle. Namely,

Corollary 1 Variational Reduction Property *The GLM equations follow from a GLM-averaged EP variational principle.*

The Variational Reduction Property is summarized by the following commutative diagram.



Sketches of Proofs. Lemma 2 follows by using the definition of GLM averaging, in combination with the four equivalence relations in the first EP theorem 8.1 and the Kelvin-Noether circulation result of the second EP theorem **??** in Appendix #1 (section 8). Three observations are relevant in sketching the proof of Lemma 2. First, one observes that the GLM average of a right-invariant Lagrangian is still right-invariant, so the Lagrange-to-Euler reduction and GLM averaging are compatible in the EP theorem. (This observation takes us along the top and down the right side of the diagram.) Second, the GLM average of the motion equation preserves the transport structure of the Kelvin circulation theorem, which is also implied by the second EP theorem. (Recall the GLM Kelvin circulation loop analysis in section 2.4.) Third, the GLM average preserves the form of the advection relations, as discussed in section 2.2. Finally, to identify the averages of thermodynamic derivatives that appear in the averaged motion equation and, thus, complete the proof, one uses commutation of exterior derivatives and GLM averaging in the definitions of these average dual variables. For example, the average temperature of a fluid parcel is correctly defined from the GLM average of the First Law, since thermodynamic relations are applied in ideal fluid theories for each fluid parcel as a *closed system*. Moreover, the *same* definitions are used in the variational derivatives of the averaged Lagrangian. These observations are sufficient to prove Lemma 2 – the EP Averaging Lemma. For more details, see Holm [2001], who proves that the EP Averaging Lemma is the front face of a cube of six interlocking equivalence relations and commutative diagrams representing the Lagrangian Averaged Euler-Poincaré (LAEP) Theorem.

Corollary 1 follows immediately from Lemma 2 for any Euler fluid

equation that is also an EP equation before the averaging is applied. However, Corollary 1 can also be proven independently, by, say, using the Clebsch procedure to place the GLM averaged equations and their average advection relations directly into the EP variational framework. Descriptions of the Clebsch procedure in this context are given in Marsden and Weinstein [1983] and in Holm and Kupershmidt [1983].

The EP Averaging Lemma and its corollary the Variational Reduction Property allow extension of the exact nonlinear GLM theory to include, for example, the continuum theory applications of the EP theorem considered in Holm, Marsden & Ratiu [1998a,b], and the geophysical fluids applications considered in Allen, Holm & Newberger [2001] and in Holm, Marsden & Ratiu [2001]. The remainder of this paper will be devoted to exploring some of the applications of the EP Averaging Lemma in the small disturbance approximation.

3.2 GLM results arising in the EP framework

The EP framework instills several fundamental properties, including some that the GLM theory is already known to possess. These known properties include the Kelvin circulation theorem, the balance laws for energy and momentum, and the potential vorticity conservation law for GLM. These properties are briefly expressed in the EP framework, as follows. For more details and the original development of the EP theory with advected parameters, see Holm, Marsden & Ratiu [1998a,b].

EP Kelvin circulation theorem for adiabatic GLM

The EP motion equation (3.1) can be rewritten in Lie-derivative form, as

$$\Big(\frac{\partial}{\partial t} + \pounds_{\bar{\mathbf{u}}^L}\Big)\Big(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\mathbf{u}^L}\cdot d\mathbf{x}\Big) = d\frac{\delta\bar{\ell}}{\delta\tilde{D}} - \frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{s}^L}d\bar{s}^L\,,$$

where $\pounds_{\bar{\mathbf{u}}^L}$ is the Lie derivative with respect to the Lagrangian mean velocity, $\bar{\mathbf{u}}^L$. Integrating this form of the EP motion equation around a loop $c(\bar{\mathbf{u}}^L)$ moving with the average motion of the fluid provides the **Kelvin-Noether theorem** in the EP framework for adiabatic compressible fluids, as

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}}^L)}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\cdot d\mathbf{x} = -\oint_{c(\bar{\mathbf{u}}^L)}\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{s}^L}\,d\bar{s}^L\,.$$

Hence, for adiabatic compressible GLM flow, from equation (3.3) for the required variational derivatives one recovers equation (2.4) as,

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}}^L)} \Big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}} + \Omega \times \mathbf{x}\Big) \cdot d\mathbf{x} = \oint_{c(\bar{\mathbf{u}}^L)} \bar{T}^L \, d\bar{s}^L \,.$$

Energy balance for adiabatic GLM

Legendre transforming in $\bar{\mathbf{u}}^L$ yields,

$$\begin{aligned} \bar{E} \ &= \ \int \frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}^L} \cdot \bar{\mathbf{u}}^L \ d^3x \ - \ \bar{\ell} \\ &= \ \int \tilde{D} \bigg[\tfrac{1}{2} |\bar{\mathbf{u}}^L|^2 + \tfrac{1}{2} \overline{|\mathbf{u}^\ell|^2} + \bar{\Phi}^L(\mathbf{x}) + \overline{e(D^\xi, \bar{s}^L)} \\ &\qquad - \overline{(\mathbf{u}^\ell + \Omega \times \xi) \cdot \partial_t \, \xi} \bigg] d^3x \,. \end{aligned}$$

Except for the last term, this is the total mean energy of the adiabatic GLM theory. We notice the "pseudoenergy"

$$\bar{e} \equiv \overline{[u^\ell_k + (\Omega \times \xi)_k] \partial_t \, \xi^k} \,, \tag{3.4}$$

appearing as the last term in the energy quantity $\bar{E}$. This term is independent of the internal energy and has a common factor with the pseudomomentum defined earlier,

$$-\,\bar{\mathbf{p}} \equiv \overline{[u^\ell_k + (\Omega \times \xi)_k] \nabla \xi^k} \,.$$

In fact, these two quantities may be expressed equivalently as

$$\tilde{D}\, \bar{e} = \overline{\pi_k \partial_t \, \xi^k} \quad \text{and} \quad \tilde{D}\, \bar{\mathbf{p}} = -\,\overline{\pi_k \nabla \xi^k} \,,$$

where $\pi_k \equiv \delta\ell / \delta(\partial_t \, \xi^k) = \tilde{D}\, [u^\ell_k + (\Omega \times \xi)_k]$ is the momentum density canonically conjugate to ξ^k, **before** the Eulerian mean is taken in the Lagrangian $\bar{\ell}$.

The spatially integrated pseudoenergy is given by

$$\langle\, \bar{e} \,\rangle = \int \tilde{D}\, \bar{e}\, d^3x = \int \overline{\pi_k \, \partial_t \, \xi^k} \, d^3x \,.$$

This term would have **cancelled**, had we performed the complete the Legendre transformation,

$$\bar{\mathcal{E}} = \int \left(\frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}^L} \cdot \bar{\mathbf{u}}^L + \overline{\frac{\delta \bar{\ell}}{\delta(\partial_t\, \xi)} \cdot \partial_t\, \xi} \right) d^3x \ - \ \bar{\ell} = \bar{E} + \int \overline{\pi \cdot \partial_t\, \xi}\, d^3x \,.$$

in both fluid and wave properties. The pseudoenergy $\bar{e}$ in equation (3.4) is thus understood to be the mean classical-mechanical action per unit mass of the fluctuating Lagrangian displacement field. This complete Legendre transformation yields the expected result for the conserved total mean energy for a self-consistent theory,

$$\bar{\mathcal{E}} = \int \tilde{D} \left[\tfrac{1}{2} |\bar{\mathbf{u}}^L|^2 + \tfrac{1}{2} \overline{|\mathbf{u}^\ell|^2} + \bar{\Phi}^L(\mathbf{x}) + \overline{e(D^\xi, \bar{s}^L)} \right] d^3x \,.$$

Hence, we find that $d\bar{E}/dt = -\frac{d}{dt} \int \overline{\pi \cdot \partial_t\, \xi}\, d^3x = -\frac{d}{dt} \int \tilde{D} \bar{e}\, d^3x$, since the total mean energy $\bar{\mathcal{E}}$ must be conserved for a theory with no sources or sinks of energy.

Remark about averaging and conservation laws. Before averaging, the integrated *instantaneous* pseudomomentum is defined as,

$$\langle \mathbf{p} \rangle = \int \tilde{D}\, \mathbf{p}\, d^3x = -\int \pi_k \nabla \xi^k\, d^3x \,,$$

The spatially integrated quantity $\langle \mathbf{p} \rangle$ generates infinitesimal Eulerian spatial shifts of the wave properties as **canonical transformations**. That is,

$$\{ \langle \mathbf{p} \rangle, \xi \} = \nabla \xi \quad \text{and} \quad \{ \langle \mathbf{p} \rangle, \pi \} = \nabla \pi \,,$$

where $\{F, H\}$ is the canonical Poisson bracket with $\{\xi(\mathbf{x}'), \pi(\mathbf{x})\} = \delta(\mathbf{x} - \mathbf{x}')$.

Under this canonical Poisson bracket, one may verify the formulas

$$\mathsf{A} = -\int \pi \cdot \partial_a \xi\, d^3x \quad \Rightarrow \quad \{\mathsf{A}, \xi\} = \partial_a \xi \quad \text{and} \quad \{\mathsf{A}, \pi\} = \partial_a \pi \,.$$

That is, the functional A generates a translation in phase space for any parameter a that admits integration by parts. If the solutions in phase space (π, ξ) are averaged over such a parameter, then the averaged generator of the the translations, $\bar{\mathsf{A}} = -\int \overline{\pi \cdot \partial_a \xi}\, d^3x$, will be conserved. For example, the i^{th} component $\bar{p}_i$ of the pseudomomentum would be conserved, if the solutions (π, ξ) were averaged over space in the i^{th} direction.

Remark – the relation between GLM and WMFI. The GLM and WMFI theories are closely related. For example, the WMFI wave action density has the same character as the GLM quantities, pseudomomentum and pseudoenergy, which may also be aptly expressed in terms of a single-frequency WKB wave packet. By varying the wave properties ξ in the averaged Lagrangian as well as the mean fluid properties, Gjaja & Holm [1996] constructed a *self-consistent* Lagrangian mean WMFI theory. This WMFI theory reduces to GLM theory when the statistics of ξ are *prescribed*. See Appendix #3 (section 10) for a brief discussion.

To explain how the wave action density of the WMFI theory is related to the GLM pseudomomentum, we make the following precanonical transformation,

$$\tilde{D}\,\bar{\mathbf{p}}\cdot d\mathbf{x} = -\,\overline{\pi_k\cdot\nabla\xi^k}\cdot d\mathbf{x} = -\,\overline{\pi\cdot d\xi}\,.$$

If ξ and π depend on a phase parameter ϕ, we may write the phase-averaged differential relation as

$$-\,\overline{\pi\cdot d\xi} = -\overline{\pi_k\partial_\phi\xi^k}\,d\phi = N d\phi = N\mathbf{k}\cdot d\mathbf{x}\,,$$

where the wavevector $\mathbf{k}$ is defined by $d\phi = \nabla\phi\cdot d\mathbf{x} = \mathbf{k}\cdot d\mathbf{x}$.

Thus, we obtain the wave action density $N = -\overline{\pi_k\partial_\phi\xi^k}$, which is related to the GLM pseudomomentum by $\tilde{D}\bar{\mathbf{p}} = N\mathbf{k}$. For the WKB wavepacket $\xi = \frac{1}{2}(\mathbf{a}e^{i\phi/\epsilon} + \mathbf{a}^*e^{-i\phi/\epsilon})$, one finds the formula,

$$\frac{N}{\tilde{D}} = -\,\overline{\Big[\frac{D^L\xi}{Dt} + (\Omega\times\xi)\Big]\cdot\partial_\phi\xi} = 2\tilde{\omega}|\mathbf{a}|^2 + 2i\Omega\cdot\mathbf{a}\times\mathbf{a}^* + 2\Im\Big(\mathbf{a}\cdot\frac{D^L\mathbf{a}^*}{Dt}\Big)\,,$$

in which $\tilde{\omega} = -D^L\phi/Dt = \omega - \mathbf{k}\cdot\bar{\mathbf{u}}^L$ is the Doppler-shifted wave frequency. This formula is in agreement with the wave action density N appearing in WMFI studies such as that of Gjaja & Holm [1996]. As a result of the symmetry under translations in ϕ introduced by phase-averaging the Lagrangian, we have

$$0 = -\,\frac{\partial}{\partial t}\frac{\partial\bar{\mathcal{L}}}{\partial(\partial_t\phi)} - \operatorname{div}\frac{\partial\bar{\mathcal{L}}}{\partial(\nabla\phi)} = \frac{\partial N}{\partial t} + \frac{\partial}{\partial x^j}\Big(N\bar{u}^{Lj} - \overline{p^\xi K_i^j\partial_\phi\,\xi^i}\Big)\,,$$

upon using the variational derivatives in equation (3.3). Andrews & McIntyre [1978b] obtain the same conservation law by directly manipulating the GLM motion equation (2.4). This equivalence, of course, is guaranteed by the EP Averaging Lemma 2.

We recover this conservation law as a result of **Noether's theorem** for the averaged Lagrangian. Thus, we have the following.

Lemma 3 *When averaging introduces an ignorable coordinate, the average of the corresponding canonically conjugate momentum is conserved. In this case, the conserved wave action density N is the phase-averaged generator of phase shifts.*

Remarks.

- The GLM pseudoenergy is related to N by $\tilde{D}\bar{e} = N\omega$, which again identifies $\bar{e}$ as an action variable.
- The self-consistent WMFI theory is closed by writing the pseudomomentum as $\tilde{D}\bar{\mathbf{p}} = N\mathbf{k}$ and using the *conservation of waves* relation, $\partial_t\,\mathbf{k} = \nabla\omega$. In this equation, the frequency variable ω must still be determined. Until this point, no small-amplitude assumption has been made. Introducing a small amplitude approximation allows the frequency ω to be determined from its dispersion relation in terms of fluid and wave mean properties. See Gjaja & Holm [1996] for more details, including the Lie-Poisson Hamiltonian structure of the self-consistent WMFI theory, which is reminiscent of the Landau two-fluid model of superfluid Helium.

EP momentum balance for adiabatic GLM

The momentum conservation law for the EP theory is,

$$\partial_t\,\bar{m}_i^L + \partial_j\,\bar{T}_i^j = \frac{\partial\bar{\mathcal{L}}}{\partial x^i}\bigg|_{exp}\,, \tag{3.5}$$

where $\bar{\mathbf{m}}^L = \delta\bar{\ell}/\delta\bar{\mathbf{u}}^L$ is the Lagrangian-mean momentum density, the stress tensor $\bar{T}_i^j$ is given by

$$\bar{T}_i^j \;=\; \bar{m}_i\bar{u}^{L\,j} + \delta_i^j\Big(\bar{\mathcal{L}} - \tilde{D}\frac{\partial\bar{\mathcal{L}}}{\partial\tilde{D}}\Big)$$

and $\partial\bar{\mathcal{L}}/\partial x^i\big|_{exp}$ denotes the derivative with respect to the explicit spatial dependence that arises in the mean Lagrangian $\bar{\ell}$ in (3.2) after

averaging over the statistics of ξ. For the adiabatic GLM theory, this stress tensor is given by

$$\bar{T}_i^j \;=\; \tilde{D}\Big(\bar{u}_i^L - \bar{p}_i + (\Omega\times\mathbf{x})_i\Big)\bar{u}^{L\,j} + \delta_i^j\,\tilde{D}\,\overline{(p^\xi/D^\xi)}\,. \tag{3.6}$$

The momentum balance law for adiabatic GLM is specified, only after $\partial\bar{\mathcal{L}}/\partial x^i\big|_{exp}$ is specified, by giving the spatial dependence in (3.2) of the wave properties and external potential in the Lagrangian density $\bar{\mathcal{L}}$. This is the requirement for obtaining closure in the GLM theory.

Local EP potential vorticity conservation for adiabatic GLM

Invariance of the Lagrangian under diffeomorphisms (interpreted physically as Lagrangian particle relabeling) implies the local conservation law for EP potential vorticity,

$$\frac{D^L}{Dt}\bar{q}^L = 0\,,\quad \text{where}\quad \bar{q}^L = \frac{1}{\tilde{D}}\nabla\bar{s}^L\cdot\text{curl}\Big(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\Big)\,.$$

For the adiabatic GLM case, the potential vorticity is given explicitly as

$$\bar{q}^L = \frac{1}{\tilde{D}}\nabla\bar{s}^L\cdot\text{curl}\Big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}} + \Omega\times\mathbf{x}\Big)\,.$$

Note the relation of the potential vorticity to the Kelvin circulation theorem. This is particularly apparent when the Kelvin theorem for adiabatic GLM theory is re-cast in terms of surface integrals using Stokes theorem, as

$$\frac{d}{dt}\iint_A \text{curl}\Big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}} + \Omega\times\mathbf{x}\Big)\cdot\hat{\boldsymbol{n}}\,dA = \iint_A \nabla\bar{T}^L\times\nabla\bar{s}^L\cdot\hat{\boldsymbol{n}}\,dA\,,$$

where the boundary of the surface A is the fluid loop, $\partial A = c(\mathbf{u}^L)$.

GLM helicity

The EP helicity is given by

$$\bar{\Lambda}^L = \int\Big(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\Big)\cdot\text{curl}\Big(\frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\Big)d^3x\,.$$

The corresponding GLM helicity is not conserved in the adiabatic case, although it is conserved in the GLM theory for the three dimensional barotropic case. (The same is true, before averaging.)

3.3 EP results for the GLM Boussinesq stratified fluid

The Eulerian expression of the averaged Lagrangian for a Boussinesq stratified fluid is

$$\boxed{\begin{aligned}\bar\ell(\bar{\mathbf{u}}^L,\tilde D,\bar\theta^L) \;&=\; \int\Big\{\tilde D\Big[\frac12\overline{\Big|\bar{\mathbf{u}}^L+\frac{D^L\xi}{Dt}\Big|^2} + \overline{(\bar{\mathbf{R}}^L+\mathbf{R}^\ell)\cdot\Big(\bar{\mathbf{u}}^L+\frac{D^L\xi}{Dt}\Big)} \\ &\qquad - \overline{\Phi(\mathbf{x}^\xi)} - g\,z\,\bar\theta^L\Big] - \overline{p^\xi\Big(\tilde D-\mathcal{J}\Big)}\Big\}d^3x\,. \quad (3.7)\end{aligned}}$$

This mean Lagrangian $\bar\ell \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}^L,\tilde D,\bar\theta^L)d^3x$ is a straight GLM decomposition of the standard Lagrangian for Boussinesq stratified fluids, followed by taking the Eulerian mean. The relative buoyancy θ is advected as a scalar in the Boussinesq approximation,

$$\partial_t\,\theta + \mathbf{u}\cdot\nabla\theta = 0\,,$$

so we have already substituted $\theta^\xi = \bar\theta^L$. The rotation frequency Ω depends on position and is given by $2\Omega(\mathbf{x}^\xi) = (\mathrm{curl}\,\mathbf{R})^\xi$. The rotation potential is decomposed in standard GLM fashion as $\mathbf{R}^\xi = \bar{\mathbf{R}}^L + \mathbf{R}^\ell$. Finally, the pressure p^ξ is a Lagrange multiplier that imposes the the constraint relation defining the conserved GLM density $\tilde D = D^\xi\mathcal{J}$, for $D^\xi = 1$.

> **Remark.** The kinetic energy is the same here as in equation (3.2) for the adiabatic compressible fluid, and the relative buoyancy is perfectly analogous to the entropy per unit mass. Moreover, the pressure constraint is also analogous to internal energy. So, one should expect no substantial difference to occur in passing from the adiabatic GLM case to the Boussinesq GLM equations.

Variational derivatives and EP equation for GLM Boussinesq stratified fluid

The variational derivatives required for the EP equation (3.1) – with entropy $\bar s^L$ replaced by buoyancy $\bar\theta^L$ – are obtained from (ignoring variational derivatives in ξ now and henceforth)

$$\delta\bar\ell = \int d^3x\,\Big[\tilde D\Big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}} + \bar{\mathbf{R}}^L\Big)\cdot\delta\bar{\mathbf{u}}^L - \;\tilde D\,gz\,\delta\bar\theta^L - \Pi^B\,\delta\tilde D\Big]\,. \quad (3.8)$$

Here, the pseudomomentum is defined by $\bar{\mathbf{p}} = -\overline{(u_j^\ell + R_j^\ell)\nabla\xi^j}$ and the Boussinesq potential Π^B is defined by

$$\Pi^B = \pi^B + gz\,\bar{\theta}^L + \bar{\mathbf{u}}^L \cdot \bar{\mathbf{R}}^L\,,$$

where

$$\pi^B = \bar{p}^L + \bar{\Phi}^L(\mathbf{x}) - \frac{1}{2}\overline{\mathbf{u}^\xi \cdot (\mathbf{u}^\xi + 2\mathbf{R}^\ell)}\,.$$

Here $\bar{p}^L = \overline{p^\xi}$ is the Lagrangian mean pressure, cf. equation (2.6) for the potential in the adiabatic compressible case. We substitute these variational derivatives into the Euler-Poincaré (EP) equation (3.1), with the analogous replacement $\bar{s}^L \to \bar{\theta}^L$, to find the following GLM motion equation for a stratified Boussinesq fluid in Cartesian coordinates,

$$\boxed{\Big[\frac{D^L}{Dt}\big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}}\big) + \big(\bar{u}_k^L - \bar{p}_k\big)\nabla\bar{u}_k^L\Big] + 2\Omega\times\bar{\mathbf{u}}^L + \nabla\,\pi^B + g\bar{\theta}^L\hat{\boldsymbol{z}} = 0\,. \quad (3.9)}$$

Remarks.

- Stratified Boussinesq fluids and adiabatic compressible fluids admit very similar forms of the EP Averaging Lemma.
- The GLM Boussinesq motion equation (3.9) is very similar to the corresponding adiabatic compressible equation (2.4). However, it is different in an important way from the corresponding equations (8.7a) and (9.1) of Andrews & McIntyre [1978a], which both contain only $D^L\bar{\mathbf{u}}^L/Dt$, instead of the combination of four terms in square brackets written here. (This error was not repeated in Bühler and McIntyre [1998] equation (9.4).) Without this correct combination of four terms, the Kelvin circulation theorem cannot be satisfied properly.

EP Kelvin circulation theorem for GLM Boussinesq stratified fluid

The EP framework provides the **Kelvin-Noether theorem** for Boussinesq stratified fluid, in the form

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}}^L)} \frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\cdot d\mathbf{x} = -\oint_{c(\bar{\mathbf{u}}^L)} \frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\bar{\theta}^L}\,d\bar{\theta}^L\,.$$

Hence, for the GLM Boussinesq stratified fluid one has,

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}}^L)}\Big(\bar{\mathbf{u}}^L-\bar{\mathbf{p}}+\bar{\mathbf{R}}^L(\mathbf{x})\Big)\cdot d\mathbf{x}=\oint_{c(\bar{\mathbf{u}}^L)} gz\,d\bar{\theta}^L\,,$$

where $\operatorname{curl}\bar{\mathbf{R}}^L(\mathbf{x}) = 2\Omega(\mathbf{x})$. If the loop $c(\bar{\mathbf{u}}^L)$ moving with the Lagrangian mean flow lies entirely on a level surface of $\bar{\theta}^L$, then the right hand side vanishes, and one recovers for this case the "generalized Charney-Drazin theorem" for transient Boussinesq internal waves, in analogy to the discussion in Andrews & McIntyre [1978a] for the adiabatic compressible case.

Momentum balance for GLM Boussinesq stratified fluid

For a mean Lagrangian density $\bar{\mathcal{L}}$, the EP theory yields the momentum balance,

$$\partial_t\,\bar{m}_i+\partial_j\,\bar{T}_i^j=\frac{\partial\bar{\mathcal{L}}}{\partial x^i}\bigg|_{exp}\,,$$

with terms defined in analogy with the compressible GLM case.

Local potential vorticity conservation for GLM Boussinesq stratified fluid

Invariance of the Lagrangian under diffeomorphisms (interpreted physically as Lagrangian particle relabeling) implies the local conservation law for EP potential vorticity,

$$\frac{D^L}{Dt}\bar{q}^L=0\,,\quad \text{where}\quad \bar{q}^L=\frac{1}{\tilde{\bar{D}}}\nabla\bar{\theta}^L\cdot\operatorname{curl}\Big(\frac{1}{\tilde{\bar{D}}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}^L}\Big)\,.$$

For the GLM case, the potential vorticity is given explicitly as

$$\bar{q}^L=\frac{1}{\tilde{\bar{D}}}\nabla\bar{\theta}^L\cdot\operatorname{curl}\Big(\bar{\mathbf{u}}^L-\bar{\mathbf{p}}+\bar{\mathbf{R}}^L(\mathbf{x})\Big)\,.$$

Again, the EP framework explains the relation of the potential vorticity to the Kelvin circulation theorem.

Other considerations in the EP framework for the GLM Boussinesq stratified fluid closely follow the developments for the GLM adiabatic fluid, modulo simple adjustments for replacing $\bar{s}^L\to\bar{\theta}^L$, in the momentum and energy balance laws, for example.

3.4 Section summary

This section shows that passing from the Euler equations for ideal compressible and incompressible fluids to the GLM equations admits the **EP Averaging Lemma**. Namely, when the Lagrangian mean is used for averaging, the EP equations for the averaged Lagrangian are identical to the averaged EP equations. Because of the EP Averaging Lemma, one finds that the Kelvin circulation theorem, the balances for energy and momentum and the local conservation law for potential vorticity arise as general features of all GLM-averaged EP fluid theories. Concepts in GLM theory such as pseudomomentum and wave action density also arise naturally as general features in the EP context.

Thus, the EP Averaging Lemma places the exact nonlinear GLM theory into the realm of **averaged Lagrangians** for Eulerian fluid mechanics in the EP framework. This framework allows further structure-preserving approximations of the GLM equations to be made using the EP variational formulation.

Being derivable in the EP framework, the GLM theory also possesses other fundamental structure that is shared by all ideal fluid theories in the EP framework. In particular, the EP framework leads to the Lie-Poisson Hamiltonian formulation for GLM theory, as well as to the potential-vorticity Casimirs associated with this Lie-Poisson bracket. In turn, this structure leads to the energy-Casimir method for characterizing equilibrium solutions of the GLM equations for ideal fluids as critical points of a constrained energy and for establishing their nonlinear Liapunov stability conditions. For an explanation of this additional structure and many applications in fluids and plasmas, see Holm, Marsden, Ratiu & Weinstein [1985].

All of these additional features are now available to the GLM theory of fluid dynamics. However, we shall forego investigating these other implications here and pass to the formulation of an approximate set of Eulerian mean equations based on a small-amplitude approximation of the GLM theory. We refer to Holm, Marsden & Ratiu [1998a,b] for detailed descriptions, derivations and basic references to other works concerning the underlying geometry associated with the EP framework for ideal fluids with advected quantities.

Remarks.

- The GLM equations may also be obtained by averaging in Hamilton's principle at constant fluid parcel label in the Lagrangian description, then transforming the result to the Eulerian description and again using the EP theory. This approach was taken in Gjaja & Holm [1996] in developing a self-consistent WMFI theory for the Boussinesq stratified case. The same approach was taken by Holm [1999] in developing nonlinear Taylor hypothesis closures (THC) for both compressible and incompressible flows. The equivalence of these other approaches to the present approach is proven by the LAEP Theorem in Holm [2001].
- Regarding stability of the GLM solutions, see Andrews & McIntyre [1978b] for discussion of a variational principle for linear evolution of small disturbances of a Lagrangian-mean flow. In this regard, see also the classical works mentioned earlier on Lagrangian fluid stability analysis and WMFI theory.

4 Linearized Eulerian/Lagrangian fluctuation relations

In principle, the GLM theory is more accurate than an Eulerian mean theory, because its scalar advection relations hold exactly, and it preserves the Euler-Poincaré (EP) structure of the original unapproximated equations. That is, being an EP theory, GLM preserves the standard ideal fluid relations for energy, momentum and potential vorticity, as well as possessing a Kelvin circulation theorem. However, the results of any Lagrangian mean theory are often difficult to interpret accurately in an Eulerian setting. In addition, the Lagrangian mean statistics themselves are affected by the mean motion at finite-amplitude Lagrangian displacement and, thus, cannot be taken as prescribed quantities. Therefore, one sees the need for an Eulerian mean counterpart to the GLM theory in the small-amplitude approximation. A theory of this type was recently initiated in Bühler & McIntyre [1998] in the context of the gravity wave parameterization problem.

In preparation for producing a variational complement to the small-amplitude GLM theory, we shall first discuss the linearized Eulerian/Lagrangian fluctuation relations.

4.1 Taylor series approximations of Eulerian fluctuations at linear order in the Lagrangian displacement ξ

In the GLM theory, the displaced fluid velocity is given by

$$\mathbf{u}(\mathbf{x}+\xi,t) = \bar{\mathbf{u}}^L(\mathbf{x},t) + \mathbf{u}^\ell(\mathbf{x},t)\,,$$

where

$$\mathbf{u}^\ell(\mathbf{x},t) = \frac{\partial}{\partial t}\xi + \bar{\mathbf{u}}^L\cdot\nabla\xi \equiv \frac{D^L\xi}{Dt}\,.$$

A Taylor series approximation shows that the Eulerian velocity fluctuation $\mathbf{u}'$ is related to the Lagrangian disturbance velocity $\mathbf{u}^\ell$, as well as the fluctuating displacement ξ and the Eulerian mean velocity $\bar{\mathbf{u}}$ at linear order in ξ by

$$\mathbf{u}^\ell = \mathbf{u}' + \xi\cdot\nabla\bar{\mathbf{u}}\,.$$

Therefore, we find the important relation at linear order,

$$\mathbf{u}'(\mathbf{x},t) = \frac{\partial}{\partial t}\xi + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}}\,. \qquad \boxed{\mathbf{u}'\text{−equation}} \tag{4.1}$$

Likewise, for a scalar quantity χ, we have the linear-order relation, $\chi^\ell = \chi' + \xi\cdot\nabla\bar{\chi}$. Consequently, we find,

$$\chi' = -\,\xi\cdot\nabla\bar{\chi}\,, \qquad \boxed{\chi'\text{−equation}} \tag{4.2}$$

for an **advected scalar** χ (since $\chi^\ell = 0$ for an advected scalar). For a conserved density, D, the linear-order Taylor approximation is

$$D^\ell = D' + \xi\cdot\nabla\bar{D} = -\bar{D}\mathrm{div}\xi\,.$$

Consequently, the Eulerian density fluctuation D' and Eulerian mean density $\bar{D}$ are related to the Lagrangian fluctuating displacement at linear order in ξ for a conserved density D by

$$D' = -\,\mathrm{div}\,(\bar{D}\xi)\,. \qquad \boxed{D'\text{−equation}} \tag{4.3}$$

The $\mathbf{u}'$ and D' equations imply

$$(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}}) = \partial_t\,(\bar{D}\xi) - \mathrm{curl}\,(\bar{\mathbf{u}}\times\bar{D}\xi)\,.$$

Taking the divergence of this relation and using the D' equation then implies the **linearized continuity equation**,

$$\partial_t\,D' = -\,\mathrm{div}\,(\bar{D}\mathbf{u}' + D'\bar{\mathbf{u}})\,. \tag{4.4}$$

Remarks.

- The $\mathbf{u}'$, χ' and D' equations are **standard** in Lagrangian stability analysis. See Friedman & Schutz [1978a] for a historical survey of the use of these linearized fluctuation relations, especially in astrophysics.

- As a consequence of the linearized continuity equation, the Eulerian mean density $\bar{D}$ satisfies the usual **continuity equation**

$$\partial_t \bar{D} + \operatorname{div}\bar{D}\bar{\mathbf{u}} = 0\,, \tag{4.5}$$

  in terms of the Eulerian mean velocity $\bar{\mathbf{u}}$.

- The $\mathbf{u}'$ equation (4.1) may also be expressed in geometrical language in terms of the ad-operator defined on the Lie algebra of vector fields. Namely, as the linear relation $\mathbf{u}' = \partial_t \xi + \mathrm{ad}_{\bar{\mathbf{u}}}\xi$. In this expression, the ad-operator is defined in terms of the commutator operation for vector fields $[\cdot\,,\cdot]$ by

$$\mathrm{ad}_{\bar{\mathbf{u}}}\xi = [\bar{\mathbf{u}}, \xi] = \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}} = -\,\pounds_\xi\,\bar{\mathbf{u}}^{\,\sharp}\,,$$

  where superscript $(\,\cdot\,)^{\sharp}$ denotes a vector field.

- In geometrical language, the Eulerian fluctuating component of any advected quantity a is given at linear approximation in ξ by

$$a' = -\,\pounds_\xi\,\bar{a}\,,$$

  where $\bar{a}$ is the Eulerian mean and $\pounds_\xi$ denotes the Lie derivative corresponding to the diffeomorphism generated by the vector field ξ, the fluctuating displacement of the Lagrangian trajectory away from its mean position $\mathbf{x}$.

- In Appendix #2 (section 9) we explain in more detail the geometric interpretations of these linear relations for the Eulerian fluctuations of a scalar, a density and the fluid velocity in terms of the Lagrangian fluctuation displacement.

- A strong connection exists between the present approach and the GLM approach to WMFI discussed in Gjaja & Holm [1996]. In that paper, attention concentrated on modeling the Lagrangian

fluid trajectory displacement fluctuation $\xi(\mathbf{x},t)$ as a WKB wave packet. See Appendix #3 (section 10) for a list of available WMFI theories based on asymptotic expansions and the WKB approximation in Hamilton's principle. Here we shall use the linear relations for D' and $\mathbf{u}'$ to derive Eulerian mean fluid equations that approximate the GLM fluid motion equations at second order in the fluctuation displacement ξ.

Frozen-in Lagrangian fluctuations. Seen as equations for ξ, the linearized fluctuation relations for D', χ' and $\mathbf{u}'$ only determine the Lagrangian fluctuating displacement up to adding an appropriate homogeneous solution determined by the initial conditions. Such homogeneous solutions are "frozen" into the mean motion, and for them D', χ' and $\mathbf{u}'$ all vanish. For example, $\bar{D}\xi = \nabla\bar{\chi}\times\nabla\psi$ implies $D'=0$ and $\chi'=0$ for any function ψ. If, in addition, the function ψ satisfies the advection relation for the mean flow, $\partial_t\psi + \bar{\mathbf{u}}\cdot\nabla\psi = 0$, then $\mathbf{u}'=0$, as well. These homogeneous solutions are known from the traditional theory of Lagrangian stability analysis for fluids. See, e.g., Friedman & Schutz [1978a] for a clear discussion of them in this context.

Thus, homogeneous solutions of the linearized fluctuation relations (the D', χ' and $\mathbf{u}'$ equations) cause no changes in the corresponding Eulerian mean fluid quantities $\bar{D}$, $\bar{\chi}$ and $\bar{\mathbf{u}}$. That is, they generate symmetries of the Eulerian mean fluid quantities. In this way, the linearized description distinguishes between **passive** Lagrangian fluctuations that are frozen-in (i.e., move with the mean flow) and **active** Lagrangian fluctuations that propagate through the fluid in a Lagrangian sense (i.e., they move through the fluid from one fluid element to the next). We shall assume that we may choose initial conditions for which the frozen-in homogeneous contributions may be set equal to zero. Being zero initially, they will remain so. Physically, this means we are choosing the mean and current positions of each Lagrangian fluid element to refer to the same Lagrangian coordinate. Fortunately, this has been our convention from the outset, so no further changes are needed. Henceforth, the fluctuating Lagrangian displacement will be understood as the inhomogeneous component of the solution. This inhomogeneous (propagating) Lagrangian fluctuation component ξ will be determined by a **Green's function method** that obtains the inhomogeneous solution for ξ uniquely from the $\mathbf{u}'$−equation. This solution will then

imply D' and χ' from the other linearized fluctuation relations. We shall give an explicit example next.

4.2 Green's function relation $\xi = G * \mathbf{u}'$

This subsection discusses the Green's function relation $\xi = G * \mathbf{u}'$ and derives the corresponding **DuHamel formula** explicitly for the special case of affine motion.

The inhomogeneous solution of the linear velocity fluctuation relation (4.1)

$$\mathbf{u}' = \partial_t \, \xi + \mathrm{ad}_{\bar{\mathbf{u}}} \, \xi = \partial_t \, \xi + \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}} \, ,$$

may be solved in terms of a Green's function G as

$$\xi = G * \mathbf{u}' \, ,$$

where $G * \mathbf{u}'$ denotes the componentwise Green's function convolution,

$$(G * \mathbf{u}')_i = \int G_i(\mathbf{x} - \mathbf{y}, t - \tau) u_i'(\mathbf{y}, \tau) \, d^3y \, d\tau \, , \quad \text{(no sum on } i) \, .$$

> **Remark.** The vector Green's function with components G_i introduces memory (depending on the history of $\bar{\mathbf{u}}$) into the solution for ξ from $\mathbf{u}'$. Note that $\mathbf{G}$ is a *deterministic* quantity.

The components of the vector Green's function G_i, $i = 1, 2, 3$, satisfy the **deterministic dual equation**,

$$\partial_t \, G_i + (\mathrm{ad}^*_{\bar{\mathbf{u}}} \, G)_i = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau) \, .$$

In tensor index notation, this dual equation is written as

$$\partial_t \, G_i + \partial_j (G_i \bar{u}^j) + G_j \partial_i \bar{u}^j = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau) \, , \quad \forall \quad i = 1, 2, 3 \, .$$

In vector notation this equation may also be expressed as

$$\Big[\partial_t \, \mathbf{G} + \mathbf{G} \, \mathrm{div} \, \bar{\mathbf{u}} \, - \, \bar{\mathbf{u}} \times \mathrm{curl} \, \mathbf{G} + \nabla (\mathbf{G} \cdot \bar{\mathbf{u}}) \Big]_i = \delta(\mathbf{x} - \mathbf{y}) \delta(t - \tau) \, .$$

The linear expression on the left side contains several elements that are familiar from the motion equation for fluids.

Remark. Geometrically, the operation ad* is defined as being the dual to the operation $-$ad under the L_2 pairing, namely,

$$\Big\langle \mathbf{G}, \mathrm{ad}_{\bar{\mathbf{u}}}\, \xi \Big\rangle = -\Big\langle \mathrm{ad}^*_{\bar{\mathbf{u}}}\, \mathbf{G}, \xi \Big\rangle = -\Big\langle \pounds_{\bar{\mathbf{u}}}\, \mathbf{G}, \xi \Big\rangle \,.$$

Thus, $\mathbf{G} \in \mathbf{g}^*$ is a one-form density – which is the geometrical object dual under L_2 pairing to vector fields. That is, the Green's functions G_i are components of a one-form density (physically, $\mathbf{G}$ is a momentum density, if ξ is a velocity). In Cartesian coordinates, $(\mathrm{ad}^*_{\bar{\mathbf{u}}}\, \mathbf{G})_i$ is written as

$$(\mathrm{ad}^*_{\bar{\mathbf{u}}}\, \mathbf{G})_i = \partial_j (G_i \bar{u}^j) + G_j \partial_i \bar{u}^j \,.$$

We shall find these formulas convenient for interpreting the geometrical meaning of the expressions arising in the manipulations to follow.

Example: Green's function for affine mean motion

We shall derive the explicit Green's function relation $\xi = G * \mathbf{u}'$ for the inhomogeneous solution of the $\mathbf{u}'-$equation in the special case that the mean Lagrangian trajectory is given by the affine relation

$$\mathbf{x}(t, \mathbf{x}_0) = F(t) \cdot \mathbf{x}_0 + \mathbf{b}(t)$$

so that $\nabla \bar{\mathbf{u}} = \dot{F}(t) F^{-1}(t)$. In this special case, the $u_i'-$equation can be written with an integrating factor as

$$F(t) \cdot \frac{d}{dt}\Big|_{\mathbf{x}_0} \Big(F^{-1}(t) \cdot \xi \Big) = \frac{d\xi}{dt}\Big|_{\mathbf{x}_0} - \dot{F} F^{-1}(t) \cdot \xi = \mathbf{u}' \,.$$

This implies an explicit **DuHamel formula** (implying memory, or history dependence) in which the inhomogeneous solution is given by

$$\xi = G * \mathbf{u}' = F(t) \cdot \int_0^t F^{-1}(\tau) \cdot \mathbf{u}'(\mathbf{x}(\tau, \mathbf{x}_0), \tau)\, d\tau \,.$$

This formula is reminiscent of the exponential integrating factor for $\nabla \bar{\mathbf{u}} = \mathit{const}$ considered in Ristorcelli [2001] for deriving tensor eddy viscosities in a turbulence model.

The full solution for the fluctuation displacement is

$$\xi = G * \mathbf{u}' + \xi^{(0)} ,$$

where $\xi^{(0)}$ is the homogeneous solution of the $\mathbf{u}'-$equation. The homogeneous solution for the affine case is simply

$$\xi^{(0)}(\mathbf{x}(t, \mathbf{x}_0), t) = F(t) \cdot \xi^{(0)}(\mathbf{x}_0) ,$$

as follows from

$$\frac{d\xi^{(0)}}{dt}\Big|_{\mathbf{x}_0} = \dot{F}(t) \cdot \xi^{(0)}(\mathbf{x}_0) = \dot{F}F^{-1}(t) \cdot \xi^{(0)}(\mathbf{x}, t) ,$$

where $\xi^{(0)}(\mathbf{x}_0)$ is the initial value of $\xi^{(0)}(\mathbf{x}, t)$. This initial value may be set to zero, if one assumes the current and mean positions of a Lagrangian trajectory refer to the same Lagrangian coordinate.

In fact, the homogeneous solution of the $\mathbf{u}'-$equation in the **general case** is the same as Cauchy's solution for the vorticity equation of an incompressible ideal fluid, namely,

$$\xi^{(0)}(\mathbf{x}, t) = F(t, \mathbf{x}_0) \cdot \xi^{(0)}(\mathbf{x}_0) ,$$

where $F(t, \mathbf{x}_0) = \partial \mathbf{x}(t, \mathbf{x}_0)/\partial \mathbf{x}_0$ is the deformation gradient, or Jacobian, of the Lagrange-to-Euler map $(t, \mathbf{x}_0) \to \mathbf{x}$.

5 Deriving $g\ell m$ – the order $O(|\xi|^2)$ GLM equations

In this section, we shall obtain a set of Eulerian-mean equations that approximate the GLM equations at second order in the displacement ξ. Following ideas familiar in Lagrangian fluid stability analysis, we shall derive these approximate equations from a variational principle based on first taking the Eulerian mean of the second-variation of the GLM Lagrangian and then using the EP formulation. Our strategy for developing this order $O(|\xi|^2)$ approximate Eulerian mean counterpart for GLM is as follows.

We base the structure-preserving approximations of the GLM theory implemented here in the EP framework on a two-step procedure. The first step linearizes the Eulerian/Lagrangian fluctuation relation. (This linearization describes how small fluctuations of a given fluid

quantity around its Eulerian mean are related to the fluctuating displacement of a Lagrangian fluid parcel trajectory around its mean position.) The second step substitutes these linearized fluctuation relations into the second-variation of the Lagrangian in Hamilton's principle. The Eulerian mean is then applied. This produces an averaged second-variation Hamilton's principle whose coefficients are Eulerian mean second moments of fluctuating Lagrangian displacements. Finally, variations are taken with respect to Eulerian mean fluid quantities and thereby one obtains the averaged motion equation in the EP framework. Thus, the first step of this procedure is reminiscent of the traditional approach in linear Lagrangian stability analysis for fluids. This traditional approach also invokes the second-variation of a fluid action principle with respect to Lagrangian displacements. However, there is a difference – the procedure here involves fluctuating linear displacements from a mean solution, not deterministic linear displacements from a steady solution as in the traditional stability analysis.

The second-variation Lagrangian contains quadratic terms in ξ, whose coefficients depend on the Eulerian mean fluid quantities. The second step of our procedure begins by taking the Eulerian mean of the second-variation Lagrangian over the quadratic moments of these fluctuating displacements, ξ. We then take variations of this averaged Lagrangian, with respect to the Eulerian mean fluid quantities, and use the EP framework. (In Lagrangian stability analysis, first, one does not average and, second, one takes variations with respect to the Lagrangian displacements, not the steady solutions.) The resulting EP equations express the back-reaction effects on the Eulerian mean motion equation of the fluctuating displacements of the Lagrangian trajectories in terms of their Eulerian second moments. This two-step procedure is performed within the EP framework for right-invariant Lagrangians that are defined on the tangent space of a group. For fluids, this is the group of diffeomorphisms representing the fluid motions, including the Lagrangian fluctuating displacements themselves. See Appendices #1 and #2 for further discussions of this point.

We summarize this two-step procedure in symbols, as follows.

Step 1

- Linearize the fluctuation relations to find equations (4.1) -

(4.3),

$$D' = -\,\mathrm{div}\bar{D}\xi\,,\ \chi' = -\,\xi\cdot\nabla\bar{\chi}\,,\ \mathbf{u}' = \partial_t\,\xi + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}}\,.$$

- Substitute these relations into the second-variation Lagrangian, to form
$$\ell'' = \int\Big[\partial_t\,\xi\cdot A\cdot\partial_t\,\xi + \partial_t\,\xi\cdot B\cdot\xi + \xi\cdot C\cdot\xi\Big]\,d^3x\,,$$
where A, B, C are matrix operators involving the mean fluid quantities **and their gradients**, i.e., the set $\{\bar{\mathbf{u}},\bar{D},\nabla\bar{\mathbf{u}},\nabla\bar{D}\}$.

Step 2

- Take the Eulerian mean to form the total mean Lagrangian
$$\bar{\ell} = \bar{\ell}_0 + \tfrac{1}{2}\overline{\ell''}\,.$$
- Derive the $g\ell m$ motion equation for barotropic compressible fluids by computing the EP equation
$$\frac{d}{dt}\frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} + \frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{u}^{\,j}}\nabla\bar{u}^{\,j} = \nabla\frac{\delta\bar{\ell}}{\delta\bar{D}}\,,$$
for the total mean Lagrangian $\bar{\ell}$ by taking its variations
$$\delta\bar{\ell} = \int\Big[\Big(\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}\Big)\cdot\delta\bar{\mathbf{u}} + \Big(\frac{\delta\bar{\ell}}{\delta\bar{D}}\Big)\delta\bar{D}\ \Big]\,d^3x\,.$$

These variational derivatives involve Eulerian means of quadratic combinations of the Lagrangian fluctuating displacement ξ, and its derivatives $\partial_t\,\xi$ and $\nabla\xi$. For example, one combination that appears is $\overline{\pi_j\nabla\xi^j}$, where $\pi = \frac{1}{2}\delta\ell''/\delta(\partial_t\,\xi) = A\cdot\partial_t\,\xi + B\cdot\xi$ is the momentum canonically conjugate to ξ. These Lagrangian quadratic statistical moments are **unknown parameters** in the $g\ell m$ equations that must be independently specified, or modeled, in closing the equations. Thus, a number of modeling decisions must be made in closing any $g\ell m$ model.

In section 6, we shall discuss the various modeling parameters required to produce a closed $g\ell m$ model. This will be done in the context of simplifying them and constructing a more manageable class of closed equations – the alpha models – obtained by using closures based on Taylor's hypothesis of frozen-in turbulence.

The equations derived from this two-step procedure – being the small-amplitude approximation of the GLM equations – are called $g\ell m$ equations. They are derived within the EP framework. These new equations describe the dynamics of Eulerian mean fluid quantities influenced by small amplitude fluctuations. Being EP equations, they still retain the properties that result from particle relabeling symmetry. In particular, the $g\ell m$ equations retain the Kelvin-Noether circulation theorem and its associated local conservation law for potential vorticity.

5.1 Expanding the Lagrangian $\ell(\mathbf{u}, D)$ in Hamilton's principle for barotropic fluids

Into the Lagrangian ℓ we substitute $\mathbf{u} = \bar{\mathbf{u}} + \epsilon\mathbf{u}'$ and $D = \bar{D} + \epsilon D'$, then truncate at quadratic order in $\mathbf{u}'$ and D' to find

$$\ell = \ell_0 + \epsilon\ell' + \frac{\epsilon^2}{2}\ell'' , \quad \text{with} \quad {}' = \frac{d}{d\epsilon}\Big|_{\epsilon=0} .$$

Variations of ℓ are given by

$$\begin{aligned} \ell' &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \ell(\bar{\mathbf{u}} + \epsilon\mathbf{u}', \bar{D} + \epsilon D') \\ &= \Big\langle \frac{\delta\ell}{\delta\bar{\mathbf{u}}} \,,\, \mathbf{u}' \Big\rangle + \Big\langle \frac{\delta\ell}{\delta\bar{D}} \,,\, D' \Big\rangle , \end{aligned}$$

where $\langle f \,,\, g\rangle = \int f\, g\, d^3x$, is the L_2 pairing. The quadratic functional ℓ'' is the second variation of the Lagrangian ℓ in the basis $\mathbf{u}'$ and D'. That is,

$$\ell'' = \Big\langle (\mathbf{u}' , D') \,,\, D^2\ell(\bar{\mathbf{u}}, \bar{D}) \cdot (\mathbf{u}' , D') \Big\rangle . \tag{5.1}$$

(This is the connection to linear Lagrangian fluid stability theory.) Note that we treat ℓ'' genuinely as a second variation; so there are no double-prime terms, such as $\mathbf{u}''$.

The **averaged Lagrangian** at second order is then

$$\bar{\ell} = \bar{\ell}_0 + \tfrac{\epsilon^2}{2}\,\overline{\ell''} , \quad \text{since } \overline{\ell'} = 0 \,, \text{ for } \overline{u'} = 0 = \overline{D'} \,.$$

Recall that the Eulerian mean satisfies the projection property, $\bar{\bar{\mathbf{u}}} = \bar{\mathbf{u}}$, and it commutes with the spatial gradient, $\overline{\nabla\mathbf{u}} = \nabla\bar{\mathbf{u}}$. On substituting

the linearized fluctuation relations for D', χ' and $\mathbf{u}'$ into $\overline{\ell''}$, we find the expected quadratic form,

$$\overline{\ell''} = \int \Big[\overline{\partial_t \xi \cdot A \cdot \partial_t \xi} + \overline{\partial_t \xi \cdot B \cdot \xi} + \overline{\xi \cdot C \cdot \xi} \Big] \, d^3x \, .$$

The A, B, C in this quadratic form are matrix operators involving the mean fluid quantities **and their gradients**, i.e., the set $\{\bar{\mathbf{u}}, \bar{D}, \nabla\bar{\mathbf{u}}, \nabla\bar{D}\}$. Consequently, after taking variations, the contribution from the mean fluctuation Lagrangian $\overline{\ell''}$ to the mean momentum in the corresponding EP equation will depend on **second-gradients** of the mean fluid quantities. The Lagrangian $\overline{\ell''}$ is a functional of the Eulerian mean quadratic moments of the Lagrangian fluctuation displacements. Consequently, the resulting EP equation will also depend parametrically on the second-order statistics of the Lagrangian fluctuations.

> **Summary.** Thus, in the EP framework for these $g\ell m$ equations we must expand the Lagrangian to second order in ξ, take its Eulerian mean, vary it with respect to $\bar{\mathbf{u}}$ and $\bar{D}$, and then model the second-order statistics of ξ in the resulting EP motion equation for $\bar{\mathbf{u}}$.

Our next steps are:

1. Compute the mean momentum of the fluctuations,

$$\overline{\mathbf{m}''} = \frac{\delta}{\delta \bar{\mathbf{u}}} \left(\tfrac{1}{2} \overline{\ell''} \right) .$$

2. Write the EP $g\ell m$ equations for total momentum

$$\bar{\mathbf{m}} = \frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}} \, , \quad \bar{\ell} = \bar{\ell}_0 + \left(\tfrac{1}{2} \overline{\ell''} \right) .$$

3. Obtain a Kelvin circulation theorem for $g\ell m$ equations from their corresponding EP equations and the Kelvin-Noether theorem for these equations.

4. Derive the $g\ell m$ energy balance by Legendre transforming $\bar{\ell}$, the averaged Lagrangian.

5. Derive the $g\ell m$ stress tensor $\bar{T}_i^j$ in the $g\ell m$ momentum balance law, $\partial_t\,\bar{m}_i + \partial_j\,\bar{T}_i^j = \partial\mathcal{L}/\partial x^i\big|_{exp}$, including the "fluctuation stresses" by invoking Noether's theorem again.

6. Use the result in section 6 to interpret the Euler$-\alpha$ model stresses, circulation and momentum in $g\ell m$ terms.

5.2 The $g\ell m$ approximations for a barotropic compressible fluid

We shall now drop any dependence of the fluid internal energy on specific entropy, here and in what follows. Thus, we shall treat only the case of a **barotropic, or isentropic, compressible fluid**. We shall evaluate the necessary variational derivatives of $\bar{\ell}$ with respect to $\bar{\mathbf{u}}$ and $\bar{D}$ by using the relations (definitions) for $\mathbf{u}'$ and D' re-derived in Appendix #2 in terms of the infinitesimal generator $\xi(\mathbf{x},t)$.

Eulerian-mean Lagrangian at order $O(|\xi|^2)$

To second order, the Eulerian-mean of the Lagrangian for a barotropic (isentropic) compressible fluid is given by

$$\boxed{\begin{aligned}\bar{\ell}(\bar{\mathbf{u}},\bar{D}) \;=\; \bar{\ell}_0 + \tfrac{1}{2}\,\overline{\ell''} &= \int\Big[\tfrac{1}{2}\bar{D}|\bar{\mathbf{u}}|^2 - \bar{D}e(\bar{D})\Big]\,d^3x \\ &+ \int\Big[\tfrac{1}{2}\bar{D}\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar{D})}{2\bar{D}}\,\overline{D'^2}\Big]\,d^3x\,.\end{aligned}} \tag{5.2}$$

For such a fluid, the equation of state defines $c^2(\bar{D})$ via

$$\frac{\partial^2}{\partial\bar{D}^2}\big(\bar{D}e(\bar{D})\big) = \frac{\partial}{\partial\bar{D}}h(\bar{D}) = \frac{c^2(\bar{D})}{\bar{D}}\,.$$

Note, before averaging, ℓ'' in equation (5.1) is the second variation of the Lagrangian $\ell(\mathbf{u},D)$ with respect to the Eulerian mean velocity and density, evaluated at the mean fluid values, $\bar{\mathbf{u}}$ and $\bar{D}$.

The variational derivatives of the mean fluctuational parts of $\bar{\ell}$ are

given by

$$
\begin{aligned}
\delta(\tfrac{1}{2}\,\overline{\ell''}) \;=\; & \int \delta\bar{\mathbf{u}}\cdot\Big[\overline{D'\mathbf{u}'}+\overline{\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}'+D'\bar{\mathbf{u}})}\Big] \\
& + \delta\bar{D}\Bigg[\tfrac{1}{2}\overline{|\mathbf{u}'|^2}+\overline{\xi\cdot\nabla(\mathbf{u}'\cdot\bar{\mathbf{u}})} \\
& - \overline{D'^2}\,\frac{\partial}{\partial\bar{D}}\Big(\frac{c^2(\bar{D})}{2\bar{D}}\Big) - \overline{\xi\cdot\nabla\Big(D'\,\frac{c^2(\bar{D})}{\bar{D}}\Big)}\;\Bigg]\,d^3x\,.
\end{aligned}
\tag{5.3}
$$

In these formulas, recall from (4.1) and (4.3) that $D' = -\,\mathrm{div}(\bar{D}\xi)$ and

$$
\mathbf{u}'(\mathbf{x},t) = \frac{\partial\xi}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}} = \partial_t\,\xi - \;\mathrm{ad}_\xi\bar{\mathbf{u}}\,.
$$

Remarks.

- **Boundary conditions** are $\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0$ and $\hat{\mathbf{n}}\cdot\xi = 0$ on the boundary.
- Recall this is the same ξ as in the GLM theory, so we will be able to make direct comparisons between $g\ell m$ and GLM after assembling the EP equations for the order $O(|\xi|^2)$ approximate theory.
- Note that after substituting the linearized approximations for the fluctuations, the mean Lagrangian and its variational derivatives now also depend on the **gradients** of mean fluid properties.

5.3 The mean fluctuation momentum

Using the geometrical notation of Appendices #1 and #2, we express the mean fluctuational momentum in various equivalent forms as

$$
\begin{aligned}
\overline{\mathbf{m}''} \;&=\; \frac{\delta}{\delta\bar{\mathbf{u}}}\Big(\tfrac{1}{2}\,\overline{\ell''}\Big) \\
&=\; \overline{D'\mathbf{u}'} + \overline{\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}'+D'\,\bar{\mathbf{u}})} \\
&=\; \bar{D}\big(\overline{\xi\cdot\nabla\mathbf{u}'}+\overline{u'_j\nabla\xi^j}\big) + \overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}} \\
&=\; \bar{D}\overline{\pounds_\xi(\mathbf{u}')^\flat} + \overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}} \\
&\equiv\; \bar{D}\big(\bar{\mathbf{u}}^S - \bar{\mathbf{p}}\big) + \overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}}\,,
\end{aligned}
$$

where superscript 'flat' $(\,\cdot\,)^\flat$ denotes a one-form and one defines

$$\bar{\mathbf{u}}^S \equiv \overline{\xi\cdot\nabla\mathbf{u}'} \quad \text{is } \boxed{\text{Stokes mean drift velocity}}$$

$$\bar{\mathbf{p}} \equiv -\,\overline{u'_j\nabla\xi^j} \quad \text{is } \boxed{\text{GLM pseudomentum}}$$

In Cartesian components the geometrical combinations $\overline{\pounds_\xi(\mathbf{u}')^\flat}$ and $\overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}}$ are expressed as

$$\big(\,\overline{\pounds_\xi(\mathbf{u}')^\flat}\,\big)_i = \overline{\xi^j u'_{i,j}} + \overline{u'_j\xi^j_{,i}} = \big(\bar{\mathbf{u}}^S - \bar{\mathbf{p}}\big)_i$$

and

$$\big(\,\overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}}\,\big)_i = \partial_j\big(\bar{u}_i\overline{D'\xi^j}\,\big) + \bar{u}_j\,\overline{D'\partial_i\xi^j}\,.$$

These are recurring combinations of terms, reappearing throughout the $g\ell m$ theory.

$g\ell m$ pseudomomentum

Before the second-variation Lagrangian for the $g\ell m$ theory is averaged, one finds the momentum canonically conjugate to ξ, given by, cf. the linearized continuity equation (4.4),

$$\pi = \delta\bar{\ell}/\delta(\partial_t\,\xi) = (\bar{D}\mathbf{u}' + D'\,\bar{\mathbf{u}})\,.$$

The corresponding pseudomomentum for the $g\ell m$ theory is then given by

$$\tilde{\mathbf{p}} = -\overline{\pi_k\nabla\xi^k} = -\overline{(\bar{D}u'_k + D'\,\bar{u}_k)\nabla\xi^k}\,.$$

Thus, our earlier discussion indicates that a WMFI version of $g\ell m$ theory would possess a conserved wave action density given by

$$N = -\overline{\pi_k\partial_\phi\xi^k} = -\overline{(\bar{D}u'_k + D'\,\bar{u}_k)\partial_\phi\xi^k}\,.$$

Mean fluctuation momentum – incompressible case

The mean fluctuation momentum takes a simpler form in the incompressible case. Recall $D' = -\,\mathrm{div}(\bar{D}\xi)$. Consequently, $\mathrm{div}\mathbf{u} = 0$ (which implies $\mathrm{div}\bar{\mathbf{u}} = 0 = \mathrm{div}\mathbf{u}'$) is consistent with setting $\bar{D} = 1$ in the mean continuity equation

$$\partial_t \bar{D} = -\,\mathrm{div}(\bar{D}\bar{\mathbf{u}})\,.$$

Also setting $\bar{D} = 1$ in the density fluctuation gives,

$$D'\Big|_{\bar{D}=1} = -\,\mathrm{div}\xi\,.$$

Taking the divergence of the $\mathbf{u}'$ equation (4.1) then yields

$$\mathrm{div}\mathbf{u}' = 0 = \partial_t\,(\mathrm{div}\xi) + \bar{\mathbf{u}}\cdot\nabla(\mathrm{div}\xi)\,.$$

So $\mathrm{div}\,\xi = 0$ is preserved, which means we may choose **initial conditions** so that $D' = 0$. Thus, D' vanishes (after taking variations) in the incompressible case, upon invoking the preserved initial conditions $\bar{D} = 1$ and $\mathrm{div}\,\xi = 0$.

Upon setting $D' = 0$ in the formulas for the incompressible case, the mean momentum may be expressed equivalently as

$$\boxed{\begin{aligned}
\bar{\mathbf{m}} = \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}\Big|_{\bar{D}=1} &= \bar{\mathbf{u}} + \overline{\mathrm{ad}^*_{\xi}\mathbf{u}'} = \bar{\mathbf{u}} + \overline{\pounds_{\xi}(\mathbf{u}')^{\flat}} \\
&= \bar{\mathbf{u}} + \overline{\xi\cdot\nabla\mathbf{u}'} + \overline{u'_j\nabla\xi^j} \\
&= \bar{\mathbf{u}} + \bar{\mathbf{u}}^S - \bar{\mathbf{p}} \\
&= \bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}')}\,. \qquad (5.4)
\end{aligned}}$$

Here the quantities $\bar{\mathbf{u}}^S$ and $\bar{\mathbf{p}}$ are the same as in the GLM theory, when rotation is absent.

Simplifications in the $g\ell m$ Lagrangian for incompressible mean flow

For incompressible mean flow, the second order Eulerian mean $g\ell m$ Lagrangian (5.2) reduces to

$$\bar{\ell}(\bar{\mathbf{u}}, \bar{D}) = \int \Big[\tfrac{1}{2}\bar{D}\Big(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}\Big) + \bar{p}(1-\bar{D})\Big]\, d^3x\,. \qquad (5.5)$$

Here we have used the GLM density equation (2.2) and truncated at second order, by enforcing $\bar{D} = 1$ with the pressure constraint. As a result, the Eulerian mean velocity $\bar{\mathbf{u}}$ satisfying the usual continuity equation (4.5) is incompressible. Thus, in the Lagrangian $\bar{\ell}$ in this case, the fluctuations contribute only to the mean $g\ell m$ kinetic energy and the Eulerian mean flow is incompressible.

5.4 $g\ell m$ results arising in the EP framework

The motion equation for barotropic $g\ell m$

For the $g\ell m$ theory in which $\bar{\ell} \equiv \int \bar{\mathcal{L}}(\bar{\mathbf{u}}, \nabla\bar{\mathbf{u}}, \bar{D}, \nabla\bar{D}; \xi(\mathbf{x},t))\, d^3x$, the EP framework yields the equations of motion,

$$\boxed{\partial_t\, \bar{m}_i + \partial_j(\bar{m}_i \bar{u}^j) + \bar{m}_j \partial_i \bar{u}^j = \bar{D}\frac{\partial}{\partial x^i}\frac{\delta\bar{\ell}}{\delta\bar{D}} \quad \text{and} \quad \partial_t\, \bar{D} + \mathrm{div}\bar{D}\bar{\mathbf{u}} = 0\,.}$$

Here the total mean momentum $\bar{m}_i$ for $g\ell m$ is defined by

$$\begin{aligned} \bar{m}_i &= \frac{\delta\bar{\ell}}{\delta\bar{u}^i} = \frac{\partial\bar{\mathcal{L}}}{\partial\bar{u}^i} - \partial_k \frac{\partial\bar{\mathcal{L}}}{\partial\bar{u}^i_{,k}} \\ &= \bar{D}\big(\bar{u}_i + \bar{u}_i^S - \bar{p}_i\big) + \Big(\overline{\mathrm{ad}^*_\xi D'\,\mathbf{u}}\Big)_i \\ &= \bar{D}\bar{u}_i + \overline{D'\,u_i'} + \Big(\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\,\bar{\mathbf{u}})\Big)_i\,, \end{aligned} \tag{5.6}$$

where $\bar{u}_i^S$ is the GLM Stokes correction and $\bar{p}_i$ is the GLM pseudomomentum given earlier. The variational derivative with respect to mean density is obtained from

$$\frac{\delta\bar{\ell}}{\delta\bar{D}} = \frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}} - \partial_k \frac{\partial\bar{\mathcal{L}}}{\partial\bar{D}_{,k}}\,.$$

Remarks:

- Note, the **linear** fluctuation relations modify the $g\ell m$ total mean momentum $\bar{m}_i$, which, however, appears in the **nonlinearity** of the EP motion equation for the $g\ell m$ theory.

- The combination of *Lagrangian mean velocity* $\bar{\mathbf{u}}^L$ and pseudomomentum $\bar{\mathbf{p}}$ appearing as $\bar{\mathbf{u}} + \bar{\mathbf{u}}^S - \bar{\mathbf{p}} = \bar{\mathbf{u}}^L - \bar{\mathbf{p}}$ in the total mean momentum for $g\ell m$ also appears in the same way in the GLM theory. Here, however, the combination also adds to $\bar{D}^{-1}\,\overline{\mathrm{ad}^*_\xi D'\,\bar{\mathbf{u}}}$.

Mean Kelvin circulation theorem for barotropic $g\ell m$

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}=\oint_{c(\bar{\mathbf{u}})}\nabla\frac{\delta\bar{\ell}}{\delta\bar{D}}\cdot d\mathbf{x}=0$$

where for $g\ell m$ one has, in the compressible barotropic case,

$$\frac{1}{\bar{D}}\frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}}=\bar{\mathbf{u}}+\bar{\mathbf{u}}^S-\bar{\mathbf{p}}+\frac{1}{\bar{D}}\,\overline{\mathrm{ad}_\xi^* D'\,\bar{\mathbf{u}}}\,.$$

In contrast, for the incompressible case, $\mathrm{div}\bar{\mathbf{u}}=0$ and one again sets $\bar{D}=1$ and $D'=0$ in this formula (after taking variations) thereby dropping the last term. Thus, in the incompressible case, the contributions to the circulation integrands are the **same** for both $g\ell m$ and GLM theories. However, the velocities of the fluid loops in the Kelvin circulation theorems are **different**. They are $\bar{\mathbf{u}}$ for $g\ell m$ and $\bar{\mathbf{u}}^L$ for GLM.

Remarks:

- When $\mathrm{curl}\,(\bar{\mathbf{u}}^S-\bar{\mathbf{p}}+\bar{D}^{-1}\,\overline{\mathrm{ad}_\xi^* D'\,\bar{\mathbf{u}}})$ vanishes, this is the Charney-Drazin "nonacceleration theorem" for $g\ell m$ for barotropic compressible fluids. See Andrews & McIntyre [1978a] for their discussion of the GLM case.

- The *Eulerian mean vorticity* due to the fluctuations in the *incompressible* case is

$$\begin{aligned}\mathrm{curl}\,(\bar{\mathbf{u}}^S-\bar{\mathbf{p}}) &= -\,\mathrm{curl}\,\overline{(\xi\times\omega')}\,,\quad \text{where}\quad \omega'=\mathrm{curl}\,\mathbf{u}'\\ &= \overline{\xi\cdot\nabla\omega'}-\overline{\omega'\cdot\nabla\xi}\\ &= \overline{\mathrm{ad}_\xi\,\omega'}\,.\end{aligned}\tag{5.7}$$

  For potential fluctuations, one sets $\omega'=0$.

Momentum balance for barotropic $g\ell m$

For a mean Lagrangian density $\bar{\mathcal{L}}$, the EP theory yields the momentum balance,

$$\partial_t\,\bar{m}_i+\partial_j\,\bar{T}_i^j=\frac{\partial\bar{\mathcal{L}}}{\partial x^i}\bigg|_{exp}\,,$$

where the total mean momentum $\bar m_i$ for barotropic $g\ell m$ is evaluated in equation (5.6). The stress tensor is defined in the EP theory for this class of Lagrangians as

$$\bar T_i^j = \bar m_i \bar u^j + \delta_i^j \Big(\bar{\mathcal{L}} - \bar D \frac{\partial \bar{\mathcal{L}}}{\partial \bar D} + \bar D \partial_k \frac{\partial \bar{\mathcal{L}}}{\partial \bar D_{,k}} \Big) - \bar u^k_{,i} \frac{\partial \bar{\mathcal{L}}}{\partial \bar u^k_{,j}} - \bar D_{,i} \frac{\partial \bar{\mathcal{L}}}{\partial \bar D_{,j}} \,. \tag{5.8}$$

Explicitly evaluating the partial derivatives of $\bar\ell$ for $g\ell m$ gives,

$$\boxed{\begin{aligned} \bar T_i^j \;&=\; \bar m_i \bar u^j + \delta_i^j \Big[p(\bar D) + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar D)}{\bar D}\,\overline{D'^2} \Big] \qquad (5.9)\\ &+ \bar D \delta_i^j \Big[-\overline{\xi\cdot\nabla(\mathbf{u}'\cdot\bar{\mathbf{u}})} + \overline{D'^2}\,\frac{\partial}{\partial \bar D}\Big(\frac{c^2(\bar D)}{2\bar D}\Big) + \overline{\xi\cdot\nabla\Big(D'\,\frac{c^2(\bar D)}{\bar D}\Big)}\ \Big] \\ &\qquad + \bar u^k_{,i}\,\overline{(D'\bar u_k + \bar D u_k')\xi^j} + \bar D_{,i}\Big(\overline{\xi^j\mathbf{u}'}\cdot\bar{\mathbf{u}} - \frac{c^2(\bar D)}{\bar D}\,\overline{D'\xi^j}\Big)\,. \end{aligned}}$$

The momentum balance law is specified, only after $\partial\bar{\mathcal{L}}/\partial x^i\big|_{exp}$ is known. This requires specifying the explicit spatial dependence in (5.2) of the wave properties and external potential in the Lagrangian density $\bar{\mathcal{L}}$ for $g\ell m$.

> **Remark.** Thus, the form of the theory is fixed – it is the EP theory. However, its manifestations and channels for expressing energy exchange are many. Even in the barotropic case, for example, there are many different contributions to the stress tensor from the Lagrangian fluctuations ξ. These contributions are primarily isotropic, including the term $\partial\bar{\mathcal{L}}/\partial x^i\big|_{exp}$.

Energy balance for barotropic $g\ell m$

A Legendre transformation gives the energy quantity for the $g\ell m$ fluid flow, namely,

$$
\begin{aligned}
\bar{E} &= \Big\langle \frac{\delta \bar{\ell}}{\delta \bar{\mathbf{u}}} \cdot \bar{\mathbf{u}} \Big\rangle - \bar{\ell}(\bar{\mathbf{u}}, \bar{D}, \xi) \\
&= \int \Big[\tfrac{1}{2}\, \bar{D} |\bar{\mathbf{u}}|^2 + \bar{D} e(\bar{D}) + \tfrac{1}{2}\, \bar{D} \overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} + \frac{c^2(\bar{D})}{2\bar{D}}\, \overline{D'^2} \Big] d^3x \\
&\quad - \int \Big(\bar{D} \overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} \; - \; \big(\overline{\mathrm{ad}^*_\xi (\bar{D}\mathbf{u}' + D'\, \bar{\mathbf{u}})} \,\big) \cdot \bar{\mathbf{u}} \Big) d^3x \\
&= \Big[E(\bar{\mathbf{u}}, \bar{D}) \; + \; \tfrac{1}{2} \overline{E''} \,\Big] \; - \; \int \overline{\Big(\frac{\delta \bar{\ell}}{\delta (\partial_t \, \xi)} \cdot \partial_t \, \xi \Big)} d^3x \,.
\end{aligned}
$$

We recognize the last integral term as $\int \overline{\pi \cdot \partial_t \, \xi}\, d^3x$, the total "pseudoenergy" for the $g\ell m$ theory. Hence, just as for the GLM theory, but now with correspondingly different definitions of terms, we find that $d\bar{E}/dt = -\frac{d}{dt} \int \overline{\pi \cdot \partial_t \, \xi}\, d^3x$, in $g\ell m$ theory, since the mean total energy must be conserved for a self-consistently coupled theory.

> **Remark.** The quantity $\frac{1}{2}\overline{E''}$ is the same as the approximately conserved expression from acoustics due to Blokhintsev [1945],
>
> $$ \tfrac{1}{2}\overline{E''} = \int \Big[\tfrac{1}{2}\, \bar{D} \overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'} \cdot \bar{\mathbf{u}} + \frac{c^2(\bar{D})}{2\bar{D}}\, \overline{D'^2} \Big] d^3x $$
>
> as discussed in Andrews & McIntyre [1978b]. Of course, this quantity is not the pseudoenergy for barotropic $g\ell m$ theory.

5.5 Remarks about $g\ell m$ closure and rapid distortion theory

- The $g\ell m$ theory linearizes the $\mathbf{u}'$ equation, so it neglects the nonlinear term $\mathrm{div}\big(\mathbf{u}'\mathbf{u}' - \overline{\mathbf{u}'\mathbf{u}'}\big)$ that appears in the $\mathbf{u}'$ equation for Reynolds turbulence closure in the Eulerian mean setting.

- Based on ideas from Lagrangian stability analysis and closely related to ideas from WMFI theory, the $g\ell m$ equations are also related to ideas from **rapid distortion theory**. See Cambon and Scott [1999] for an interesting discussion of the close connections

between rapid distortion theory and WKB stability theory. Some of the relations between WKB stability and WMFI theory are also briefly discussed in the EP framework in Appendix #3 (section 10).

- In principle, the Lagrangian statistics for the coefficients in the nonlinear $g\ell m$ equations may be closed at second moments, since the fluctuations ξ and $\mathbf{u}'$ are both taken to satisfy *linear* equations. Thus, the linearity of the $g\ell m$ equations would allow one to derive a set of equations for second moments such as $\overline{\xi \times \operatorname{curl} \mathbf{u}'}$ in the incompressible case and treat the combined system for the motion and the Lagrangian statistics as an initial value problem. This could be done by computing $\partial_t(\overline{\xi \times \operatorname{curl} \mathbf{u}'})$ using the linearized Euler motion equation for evolving $\mathbf{u}'$ and using the $\mathbf{u}'-$equation for the evolution of ξ.

  Of course, such a linear closure would not produce only linear effects in the mean motion equation. The $g\ell m$ effects arise from second moments. The $g\ell m$ effects appear multiplicatively in the stress tensor and additively in the definition of the total mean momentum. The latter appears also in the nonlinearity of the $g\ell m$ equations. Thus, although the the mean advection relations are enforced only at *linear* order, the contributions of the fluctuations to the $g\ell m$ motion equation are both linear and nonlinear.

- We note that the $g\ell m$ theory expresses wave properties in terms of Lagrangian displacement statistics and gradients of mean flow properties. This idea suggests we may consider substituting aspects of these relations between wave properties and mean gradients, before taking variations in Hamilton's principle, by regarding these wave, mean flow relations as a type of **Taylor hypothesis**. We shall follow this idea further in section 6.

- The Green's function relation $\xi = G * \mathbf{u}'$ implies an explicit DuHamel formula (with its memory, or history dependence) and suggests that $\overline{\xi\xi}$ might be usefully computed as a **diagnostic** in direct numerical simulations. We shall follow this suggestion in the next subsection.

5.6 Determining Lagrangian fluctuation statistics in DNS

In assessing and benchmarking the $g\ell m$ model, one may obtain $\overline{\xi\xi}$ by measuring $\mathbf{u}'$ and D' and then **inverting** their Eulerian relations with displacement fluctuation ξ in direct numerical simulations. For example, in compressible simulations, the density fluctuations satisfy equation (4.3)

$$D'(\mathbf{x},t) = -\operatorname{div}(\bar{D}\xi)\,.$$

The Helmholtz decomposition $\bar{D}\xi = \nabla\phi + \operatorname{curl}\mathbf{A}$ then implies the curl-free part of $\bar{D}\xi$ as

$$\bar{D}\xi = -\nabla\Delta^{-1}D'\,.$$

The divergence-free part of $\bar{D}\xi$ (the homogeneous solution) is preserved and, so, it may be set to zero as an initialization condition. For the velocity fluctuations, we have the standard linearized relation (4.1)

$$\mathbf{u}'(\mathbf{x},t) = \frac{\partial\xi}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\xi - \xi\cdot\nabla\bar{\mathbf{u}}\,,$$

whose inhomogeneous solution is found using the Green's function inversion,

$$\xi = G * \mathbf{u}'\,.$$

As discussed earlier in section 4, this also has a homogeneous solution satisfying $d\xi^{(0)}/dt = \xi^{(0)}\cdot\nabla\bar{\mathbf{u}}$. When this solution is initialized at zero, the remaining inhomogeneous part of the solution for ξ may be determined by the Green's function method. Comparing the solutions for ξ that are found from the $D'-$ and $\mathbf{u}'-$equations may also be used as a check of the accuracy of the method, provided the numerical scheme does not unduly excite the homogeneous solutions. If this excitation does occur, then perhaps an occasional re-initialization may be required to suppress the homogeneous solutions for the Lagrangian fluctuation, ξ. This idea is reminiscent of the "slow manifold" concept, introduced in Leith [1980] for atmospheric dynamics, which initializes gravity wave amplitudes to zero. These gravity waves may need to be periodically reset to zero, especially when updating the forecast with new data. However, the homogeneous solutions in the present case do

not grow unstably as the gravity waves do in atmospheric dynamics, because these homogeneous solutions are not unstable. In fact, being frozen into the motion, they cannot grow any larger than their initial values.

In principle, performing this inversion would give us an Eulerian diagnostic for determining the Lagrangian fluctuation statistics and testing the basis for the derivation of the $g\ell m$ equations, as outlined below.

5.7 Protocol for using DNS to diagnose Lagrangian statistics input for $g\ell m$ in barotropic compressible fluid dynamics

For using DNS to diagnose the Lagrangian fluctuation statistics of $g\ell m$ in barotropic compressible fluid dynamics (or, shallow water dynamics), one could proceed as follows.

1. Measure $\bar{\mathbf{u}}$, $\mathbf{u}'$, $\bar{D}$, D', with $\overline{D'} = 0 = \overline{\mathbf{u}'}$ in the DNS.

2. Recall $\mathbf{u}' = \partial_t \xi - \mathrm{ad}_{\bar{\mathbf{u}}}\xi$, so (componentwise)
$$\xi_i = (G * \mathbf{u}')_i = \int G_i(\mathbf{x}-\mathbf{y}, t-\tau) u_i'(\mathbf{y},\tau)\, d^3y\, d\tau\,,$$
with a **deterministic** vector Green's function, whose components G_i, $i = 1, 2, 3$, each satisfy
$$\partial_t G_i + (\mathrm{ad}^*_{\bar{\mathbf{u}}} G)_i = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)\,.$$

3. In a local spatial domain, compute G, then compute $\xi = G * \mathbf{u}'$.

4. Compute $D' = -\mathrm{div}(\bar{D}\xi)$ from the result and compare with the measured D'. This checks the ($\tilde{D}-$weighted) curl-free part of the first computation of ξ against the DNS "measurement."

5. Compute $\bar{D}\xi = -\nabla\Delta^{-1}D'$. This checks the two computations of ξ against each other.

6. The result $\bar{D}\xi = -\nabla\Delta^{-1}D'$ should be curl-free. Check this, by computing $\mathrm{curl}\bar{D}\xi$.

7. Compute $\overline{\xi\xi}(\mathbf{x}, t)$. Determine the spatial and temporal dependence of its isotropic and anisotropic components.

8. Compute $\overline{\xi \mathbf{u}'}(\mathbf{x},t)$. This determines the relative dispersion tensor for Lagrangian trajectories.
9. Compute $\overline{\mathbf{m}''} = \overline{D'\mathbf{u}'} + \overline{\mathrm{ad}^*_\xi(\bar{D}\mathbf{u}' + D'\,\bar{\mathbf{u}})}$, the mean momentum due to the fluctuations.
10. Compare $\bar{\mathbf{m}}_{fluct}$ with $-\alpha^2\Delta\bar{\mathbf{u}}$. (This would extract α^2 in the isotropic case. The original derivation of the alpha model in Holm, Marsden & Ratiu [1998a,b] for the incompressible case assumes that $\overline{\xi\xi}$ is homogeneous and isotropic. See section 6 for more discussion of the alpha models.)

5.8 EP $g\ell m$ equations for incompressible mean flow

The variational derivatives of the $g\ell m$ Lagrangian (5.5) for incompressible flow

$$\bar{\ell}(\bar{\mathbf{u}},\bar{D}) = \int \Big[\tfrac{1}{2}\bar{D}\Big(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}\Big) + \bar{p}(1-\bar{D})\Big]\, d^3x\,, \tag{5.10}$$

are given by, cf. equation (5.4),

$$\begin{aligned} \delta\bar{\ell}(\bar{\mathbf{u}},\bar{D}) \;=\; & \int \Big[\delta\bar{D}\Big(\tfrac{1}{2}\big(\,|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}\,\big) - \bar{p}\Big) + \delta\bar{p}\,(1-\bar{D}) \\ & + \bar{D}\,\delta\bar{\mathbf{u}}\cdot\Big(\bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}'\,)}\Big) + \delta\bar{\mathbf{u}}\cdot\overline{\mathbf{u}'\,\mathrm{div}\,(\bar{D}\xi)}\Big]\, d^3x\,. \end{aligned}$$

We define the $g\ell m$ circulation velocity as,

$$\bar{\mathbf{v}} \equiv \bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}'\,)}\,.$$

The corresponding EP motion equation (with $\nabla\cdot\bar{\mathbf{u}}=0$) is expressed as,

$$\boxed{\frac{\partial}{\partial t}\bar{\mathbf{v}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{v}} + \bar{v}_j\nabla\bar{u}^j + \nabla\bar{p} = 0\,.}$$

This is the EP equation for the Lagrangian (5.10). It also has the equivalent form,

$$\frac{\partial}{\partial t}\bar{\mathbf{v}} - \bar{\mathbf{u}}\times\mathrm{curl}\,\bar{\mathbf{v}} + \nabla\big(\bar{\mathbf{v}}\cdot\bar{\mathbf{u}} + \bar{p}\,\big) = 0\,.$$

The Kelvin circulation theorem for the incompressible $g\ell m$ equations is simply,

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}})}\big(\bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'}\,\big)\cdot d\mathbf{x} = 0\,. \tag{5.11}$$

Remark. We recall that $\mathbf{u}' = \partial_t \xi + \bar{\mathbf{u}} \cdot \nabla\xi - \xi \cdot \nabla\bar{\mathbf{u}}$. For the case that $\nabla \cdot \bar{\mathbf{u}} = 0$ and $\nabla \cdot \xi = 0$, this becomes $\mathbf{u}' = \partial_t \xi - \operatorname{curl}\big(\bar{\mathbf{u}} \times \xi\big)$. Thus, to **close** the $g\ell m$ EP motion equation for incompressible Eulerian mean flow, only **one key element** from the Lagrangian statistics must be specified. Namely, the quantity

$$\overline{\xi \times \omega'} = \overline{\xi \times \operatorname{curl}\mathbf{u}'} = \overline{\xi \times \operatorname{curl}\Big(\partial_t \xi - \operatorname{curl}\big(\bar{\mathbf{u}} \times \xi\big)\Big)}\,, \tag{5.12}$$

must be specified in terms of $\bar{\mathbf{u}}$, $\nabla\bar{\mathbf{u}}$ and $\nabla\nabla\bar{\mathbf{u}}$. This specification is one of the objectives of the discussions in the next section.

6 Alpha models

6.1 Opening remarks

We have seen that the use of Taylor expansions in the linearized fluctuation relations summons gradients of Eulerian mean fluid quantities into the mean second-variation Lagrangian. In turn, these gradients summon **second-order** spatial derivatives such as $\nabla\nabla\bar{\mathbf{u}}$ into the $g\ell m$ motion equation that results from the EP variational principle.

Among other things, the $g\ell m$ stress tensor (5.9) for a compressible fluid shows the variety of channels available for energy exchange to occur. These channels arise through the various combinations of Eulerian mean gradients that appear in the stress tensor for the $g\ell m$ theory. The incompressible case is more straightforward because it has fewer such channels. However, to achieve closure, even the incompressible $g\ell m$ case still requires an assumption to express the key element of the Lagrangian statistics (5.12) in terms of $\bar{\mathbf{u}}$, $\nabla\bar{\mathbf{u}}$ and $\nabla\nabla\bar{\mathbf{u}}$. The linearized fluctuation equations themselves (relating the Eulerian and Lagrangian small disturbances) shall guide the formulation of such approximate closure assumptions.

Approach. The approach to the alpha-model equations is closely related to the $g\ell m$ approach, but with one important difference. Namely, the order is interchanged in the steps of making approximations and varying the EP Lagrangian in Hamilton's principle.

To obtain the $g\ell m$ equations:
We Expanded the Lagrangian, Took its Eulerian mean, then Varied to obtain the equations of motion, and finally saw the need to Approximate the closure. This could be done, in principle, by using the $\mathbf{u}'-$equation for the tendency of ξ and the linearization of the GLM equations for the tendency of $\mathbf{u}'$. We shall discuss a more direct approach to closure.

To obtain the $\alpha-$models:
We shall Expand the Lagrangian, Take its Eulerian mean, Approximate in the Lagrangian (by taking a particular solution of the $\mathbf{u}'-$equation as a **Taylor hypothesis**), and then Vary to find a closed set of EP motion equations.

Remark. Because of the close relation between the approaches used in deriving these two sets of equations, one might hope for a bridge between them. For example, the $g\ell m$ equations could potentially provide an Eulerian diagnostic for determining parameters in the alpha model from DNS of the full Euler equations (or Navier-Stokes equations). The $g\ell m$ equations form a systematic approximation for the original GLM equations, within the EP framework. Thus, perhaps the GLM equations could be used to help answer questions that may arise at the other levels of approximation in this framework, particularly, in the alpha models.

6.2 Taylor hypothesis closure (THC) approach

We shall use partial, or particular, solutions of the linearized velocity fluctuation equation (4.1)

$$\mathbf{u}' = \partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}} \,,$$

to guide certain choices of **Taylor hypotheses**. The three Taylor hypotheses we shall discuss for the $\mathbf{u}'-$equation are:[3]

THC#1 Neglect space and time derivatives of ξ, or, set $\partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi = 0$. In the incompressible case, this yields the original Euler-alpha

[3]These three Taylor hypothesis closures are for the $g\ell m$ equations. Later, we shall mention THC#4 – for the GLM equations.

model of Holm, Marsden & Ratiu [1998a,b], in which one assumes $\mathbf{u}' = -\,\xi \cdot \nabla \bar{\mathbf{u}}$.

THC#2 Assume that ξ is frozen-in as a one-form. In the incompressible case, this yields the anisotropic alpha model of Marsden & Shkoller [2001], with $\mathbf{u}' = -2\xi \cdot \bar{\mathsf{e}}$, where $\bar{\mathsf{e}} = \frac{1}{2}(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$ is the mean strain rate tensor, and $\partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi = -\nabla \bar{\mathbf{u}}^T \cdot \xi$.

THC#3 Assume that ξ is frozen-in as a two-form in three dimensions. Hence, set $\partial_t \xi + \bar{\mathbf{u}} \cdot \nabla \xi = \xi \cdot \nabla \bar{\mathbf{u}} - \xi \operatorname{div} \bar{\mathbf{u}}$. This assumption implies $\mathbf{u}' = -\,\xi \operatorname{div} \bar{\mathbf{u}}$, which, of course, is only interesting in the compressible case. For compressible flows, this choice lead to similarities with the Green-Naghdi equation for shallow water dynamics. The Green-Naghdi equation is discussed in Green & Naghdi [1976]. See, e.g., Camassa, Holm & Levermore [1997] for more references and an asymptotic treatment of these equations.

Remark about algebraic closure. Neglecting the partial time derivative in the linearized velocity fluctuation equation for $\mathbf{u}'$ leads to an "algebraic closure relation," expressed as,

$$
\mathbf{u}' = \bar{\mathbf{u}} \cdot \nabla \xi - \xi \cdot \nabla \bar{\mathbf{u}} = \operatorname{curl}\left(\xi \times \bar{\mathbf{u}}\right),
$$

in the incompressible case. We note that substituting this algebraic closure relation into the $g\ell m$ Lagrangian (5.2) yields the following contribution to the mean fluctuational momentum,

$$
\overline{\mathbf{m}''} = \frac{\delta}{\delta \bar{\mathbf{u}}} \int \tfrac{1}{2}\, \overline{|\mathbf{u}'|^2}\, d^3x = -\,\overline{\xi \times \operatorname{curl}\operatorname{curl}\left(\xi \times \bar{\mathbf{u}}\right)}\,.
$$

So, in this incompressible case, the contribution of the fluctuations to the mean total momentum (or pseudomomentum) keeps the same $g\ell m$ form as in equations (5.11) and (5.12). However, this expression is not closed, because it still remains to specify the evolution of the Lagrangian statistics. This could perhaps be accomplished by imposing the algebraic closure relation as a constraint in the Lagrangian. The problem does not arise in using the Taylor hypotheses THC#1–#3, above.

The Taylor hypotheses THC#1–#3 are approximate relations between Eulerian and Lagrangian statistics (namely, they are relations for $\mathbf{u}'$ as a function of ξ and its derivatives) that yield closures when substituted into the averaged Lagrangian in Hamilton's principle. We shall first discuss the incompressible case, which is simpler, and then we shall discuss the barotropic compressible case.

In both cases we shall illustrate the **Taylor hypothesis closure technique** by substituting the first of these three Taylor hypotheses into the $g\ell m$ Lagrangian, before taking its variations. This approach results via the EP framework in closed equations based on the $g\ell m$ equations that retain their Kelvin circulation theorem and conservation properties. Among these closed equations for the incompressible case are variants of the Euler-alpha model (or, averaged Euler equations) that are also related to the theory of second grade fluids and have been discussed as potential turbulence closure models when Navier-Stokes viscous dissipation is introduced, as in Chen et al. [1998], [1999a,b,c]. We shall show how this approach also leads to a new generalization of the Euler-alpha model that includes compressibility.

6.3 A brief history of the alpha models

The Euler-alpha equations for averaged incompressible ideal fluid motion were first derived in Holm, Marsden & Ratiu [1998a] in the context of the Euler-Poincaré theory for fluid dynamics. That derivation proceeded essentially by choosing the kinetic energy to be the H_1 norm of the Eulerian fluid velocity, rather than the usual L_2 norm. This choice generalized the unidirectional shallow water equation of Camassa and Holm [1993] from one dimension to three dimensions. The resulting n-dimensional Euler-alpha equation is (with $\nabla\cdot\mathbf{u}=0$, $\mathbf{v}\equiv\mathbf{u}-\alpha^2\Delta\mathbf{u}$ and constant length scale α)

$$\frac{\partial}{\partial t}\mathbf{v}+\mathbf{u}\cdot\nabla\mathbf{v}+v_j\nabla u^j+\nabla p=0\,. \tag{6.1}$$

This is the EP equation for the Lagrangian

$$\ell=\tfrac{1}{2}\int|\mathbf{u}|^2+\alpha^2|\nabla\mathbf{u}|^2\;d^3x\,, \tag{6.2}$$

for a constant α and divergenceless fluid velocity $\mathbf{u}$. Mathematically, this equation describes geodesic motion on the volume-preserving diffeomorphism group of $\mathbf{R}^3$ relative to the H_1 norm in a sense similar

to the work of Arnold [1966] and Ebin and Marsden [1970] in which the Euler equations are shown to describe geodesic motion on the same diffeomorphism group relative to the L_2 kinetic energy norm.

Remarkably, the H_1-geodesic Euler-alpha equation was later recognized as being identical to the well-known invisicid second grade fluid equations introduced by Rivlin and Ericksen [1955], although of course these equations were derived from a completely different viewpoint. The differences in their derivations imply corresponding differences in the interpretations of the solutions of these equations in each of their contexts. In particular, the constant parameter alpha (a length scale) is interpreted differently in the two theories. In the Euler-alpha model, the parameter alpha is associated with the flow regime and, in numerical simulations, **alpha separates active and passive degrees of freedom**, as shown in Chen et al. [1999c]. (Physically, alpha is the smallest active length scale participating in the nonlinear interactions – so scales smaller than alpha are swept along by the larger ones.) In contrast, for the theory of second grade fluids, alpha is a thermodynamic material parameter.

Extensions of Euler$-\alpha$. The works of Marsden & Shkoller [2001] and Marsden, Ratiu & Shkoller [2001] used the EP framework to introduce a certain type of filtering – called "fuzzying" – into the Lagrangian. Applying the EP reduction theorem to the Lagrangian for "fuzzy flow" yielded an alternative formal derivation of the incompressible Euler$-\alpha$ model, as well as an anisotropic variant of it and the extension of that variant to Riemannian manifolds. These works also showed short time existence for solutions of the Euler$-\alpha$ model and these extensions, by establishing for it the analog of the Ebin-Marsden theorem for the incompressible Euler equations, proven in Ebin and Marsden [1970]. See also Shkoller [1998] for the corresponding existence result for the original Euler-alpha model of Holm, Marsden & Ratiu [1998a].

While these references do make use of the EP reduction theorem, they do not show that the reduced equations obtained from it would also result from applying the "fuzzy flow" averaging method directly to Euler's equations. The EP Averaging Lemma *guarantees* this result, however, when the GLM averaging method is applied. For example, to the extent that the fuzzy averaging used in Marsden and Shkoller [2001] fails to possess the projection property, the resulting EP equations will

fail to coincide with fuzzy average of the original equations. Investigations of the relation between fuzzy averaging and GLM averaging are underway.

THC# 4, a nonlinear GLM Taylor hypothesis. In Holm [1999], a nonlinear GLM Taylor hypothesis was introduced and applied for both compressible and incompressible flows. This Taylor hypothesis THC#4 for GLM assumes that the Lagrangian displacement fluctuation ξ is frozen as a *Lagrangian vector field* into the *nonlinear* GLM flow. Namely, (note $\bar{\mathbf{u}}^L$ rather than $\bar{\mathbf{u}}$)

$$\mathbf{u}^\ell \equiv \partial_t \, \xi + \bar{\mathbf{u}}^L \cdot \nabla \xi = \xi \cdot \nabla \bar{\mathbf{u}}^L \, . \tag{6.3}$$

This THC#4 is substituted directly into the GLM averaged Lagrangian, e.g., (3.2), or (3.7), *without* linearizing the fluctuation relations. One may then vary the Lagrangian in the EP framework, *without* making the small-amplitude approximation. This THC#4 treats the fluctuating Lagrangian displacement ξ as a material property associated with the frozen-in GLM motion of a "cloud" of fluid parcels initially displaced from one another by the initial value of ξ, which is *not* taken to vanish in this case. Under the GLM dynamics, the assumed nonlinear frozen-in relation for THC#4 implies additional flow stresses as each fluid parcel convects this material property.

The Taylor hypothesis closure THC#4 may appear formally similar to the others, especially to THC#1. However, THC#4 differs fundamentally from the others by being imposed as a finite, rather than a small-amplitude, approximation. Thus, THC#4 couples to the graients of the full Lagrangian mean velocity, rather than to the gradients of its Eulerian mean small-amplitude approximation. Of course, the other Taylor hypothesis closures THC#1–#3 could also be made at the nonlinear GLM level, without first making the linearized fluctuation hypotheses that lead to the $g\ell m$ theory. It turns out that THC#1 leads to a trivial result in this case, and the other nonlinear Taylor hypotheses have not yet been analyzed at the GLM level. The implications and physical interpretations of the GLM results of THC#4 are discussed in Holm [1999]. This includes discussions of an interesting duality between the Eulerian-mean and Lagrangian-mean fluid velocities that arises for THC#4 in the GLM theory.

6.4 Barotropic $\overline{g\ell m}$, a closure model for barotropic $g\ell m$

In seeking its variational closure, we shall start with the small amplitude $g\ell m$ Lagrangian (5.2) for barotropic compressible flow at second order,

$$\begin{aligned}\bar\ell &= \bar\ell_0 + \tfrac{1}{2}\,\overline{\ell''} = \int \Big[\tfrac{1}{2}\bar D|\bar{\mathbf{u}}|^2 - \bar D e(\bar D)\Big)\Big]\, d^3x \\ &\qquad + \int \Big[\tfrac{1}{2}\bar D\overline{|\mathbf{u}'|^2} + \overline{D'\mathbf{u}'}\cdot\bar{\mathbf{u}} - \tfrac{1}{2}\tfrac{c^2(\bar D)}{\bar D}\,\overline{D'^2}\Big]\, d^3x\,. \end{aligned} \tag{6.4}$$

Into this $g\ell m$ Lagrangian we shall substitute the simplest available hypothesis for closing the barotropic $g\ell m$ system, namely[4]

$$\mathbf{u}' = -\,\xi\cdot\nabla\bar{\mathbf{u}} \quad\text{and}\quad D' = -\,\xi\cdot\nabla\bar D\,. \tag{6.5}$$

This substitution yields the mean Lagrangian for the closed barotropic $g\ell m$ system (barotropic $\overline{g\ell m}$)

$$\begin{aligned}\bar\ell &= \bar\ell_0 + \tfrac{1}{2}\,\overline{\ell''} = \int \Big[\tfrac{1}{2}\bar D|\bar{\mathbf{u}}|^2 - \bar D e(\bar D)\Big)\Big]\, d^3x \\ &\qquad + \int \Big[\tfrac{1}{2}\bar D\,\overline{\xi^k\xi^l}\,\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \overline{\xi^k\xi^l}\,\bar{\mathbf{u}}\cdot\bar{\mathbf{u}}_{,k}\,\bar D_{,l} - \tfrac{c^2(\bar D)}{2\bar D}\,\overline{\xi^k\xi^l}\,\bar D_{,k}\bar D_{,l}\Big]\, d^3x\,. \end{aligned} \tag{6.6}$$

With the $\overline{g\ell m}$ closure hypothesis (6.5) no derivatives of the fluctuation statistics appear in this mean Lagrangian.

Combining the closure hypothesis (6.5) with the $\mathbf{u}'$–equation (4.1) implies $(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\xi = 0$, i.e., componentwise advection of ξ. Consequently, the components of the quadratic Lagrangian moments are simply carried along with the Eulerian mean flow, as

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\,\overline{\xi^k\xi^l} = 0\,.$$

This equation admits the isotropic solution

$$\overline{\xi^k\xi^l} = \alpha^2\,\delta^{kl}\,,$$

where α is an advected scalar, $(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\alpha = 0$, that has dimensions of length. In turn, this advective relation for α also admits a constant

[4]For compressible flows, the other Taylor hypotheses THC#2 & THC#3 lead to similar formulas to those given in this section. We shall not discuss those other cases here. The implications of Taylor hypothesis THC#2 for *incompressible* flows are analyzed in Marsden and Shkoller [2001].

solution, should we wish to simplify the dynamics of the Lagrangian moments even further.

To develop the barotropic closure model $\overline{g\ell m}$, we shall substitute the variational derivatives of $\bar{\ell}$ into the following EP equation, cf. equation (3.1),

$$\boxed{\frac{\partial}{\partial t}\frac{\delta\bar{\ell}}{\delta\bar{u}^i} + \frac{\partial}{\partial x^k}\Big(\frac{\delta\bar{\ell}}{\delta\bar{u}^i}\bar{u}^k\Big) + \frac{\delta\bar{\ell}}{\delta\bar{u}^k}\frac{\partial\bar{u}^k}{\partial x^i} = \bar{D}\frac{\partial}{\partial x^i}\frac{\delta\bar{\ell}}{\delta\bar{D}} - \frac{\delta\bar{\ell}}{\delta\overline{\xi^k\xi^l}}\frac{\partial\overline{\xi^k\xi^l}}{\partial x^i}\,. \qquad (6.7)}$$

The contribution of the last term arises from the scalar advection of the components of $\overline{\xi^k\xi^l}$. The necessary variational derivatives may be obtained from

$$\delta\bar{\ell} \;=\; \int\bigg[\Big((1-\hat{\Delta})(\bar{D}\bar{\mathbf{u}}) + \overline{\xi^k\xi^l}\,\bar{\mathbf{u}}_{,k}\bar{D}_{,l}\Big)\cdot\delta\bar{\mathbf{u}} \qquad (6.8)$$
$$+\Gamma_{k\,l}\delta(\,\overline{\xi^k\xi^l}\,) - \bar{\Pi}^{g\ell m}\delta\bar{D}\bigg]d^3x\,,$$

with homogeneous boundary conditions,

$$\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0 \quad\text{and}\quad \hat{\mathbf{n}}\cdot\overline{\xi\xi} = 0 \quad\text{on the boundary.}$$

These are the physically meaningful conditions at fixed boundaries. Weaker boundary conditions may also suffice in this case, namely,

$$\hat{\mathbf{n}}\cdot\bar{\mathbf{u}} = 0 \quad\text{and}\quad \hat{\mathbf{n}}\times\big(\hat{\mathbf{n}}\cdot\overline{\xi\xi}\cdot\nabla\big)\bar{\mathbf{u}} = 0 \quad\text{on the boundary.}$$

Here the generalized Laplacian operator $\hat{\Delta}$ is defined by

$$\hat{\Delta} = \partial_l\,\overline{\xi^k\xi^l}\,\partial_k\,, \qquad (6.9)$$

and the $\overline{g\ell m}$ potential $\bar{\Pi}^{g\ell m}$ is defined by

$$\bar{\Pi}^{g\ell m} = (1-\hat{\Delta})\big(\tfrac{1}{2}|\bar{\mathbf{u}}|^2 - h(\bar{D})\big) + \tfrac{1}{2}\,\overline{\xi^k\xi^l}\,\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} - \tfrac{1}{2}\,\overline{\xi^k\xi^l}\,\bar{D}_{,k}\bar{D}_{,l}h''(\bar{D})\,,$$

where $h'(\bar{D}) = c^2(\bar{D})/\bar{D}$. The quantity $\Gamma_{k\,l}$ denotes the variational derivative of $\bar{\ell}$ with respect to the mean Lagrangian statistical moments. Namely,

$$\Gamma_{k\,l} = \frac{\delta\bar{\ell}}{\delta\,\overline{\xi^k\xi^l}} = \tfrac{1}{2}\,\bar{D}\,\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \bar{\mathbf{u}}\cdot\bar{\mathbf{u}}_{,k}\,\bar{D}_{,l} - \tfrac{1}{2}\,\bar{D}_{,k}\bar{D}_{,l}\,h'(\bar{D})\,.$$

The EP motion equation (6.7) for barotropic $\overline{g\ell m}$ is, thus,

$$\boxed{\partial_t \bar{m}_i + \partial_j(\bar{m}_i\bar{u}^j) + \bar{m}_j\partial_i\bar{u}^j = \bar{D}\,\partial_i\,\bar{\Pi}^{g\ell m} - \Gamma_{k\,l}\,\partial_i\,\overline{\xi^k\xi^l}\,, \qquad (6.10)}$$

where the total mean momentum for barotropic $\overline{g\ell m}$ is given by

$$\begin{aligned}\bar{\mathbf{m}} = \frac{\delta\bar{\ell}}{\delta\bar{\mathbf{u}}} &= (1-\hat{\Delta})(\bar{D}\bar{\mathbf{u}}) + \overline{\xi^k\xi^l}\,\bar{\mathbf{u}}_{,k}\,\bar{D}_{,l} \\ &= \bar{D}\bar{\mathbf{u}} - \tfrac{1}{2}\left[\hat{\Delta}(\bar{D}\bar{\mathbf{u}}) + \bar{D}\hat{\Delta}\bar{\mathbf{u}} + \bar{\mathbf{u}}\,\hat{\Delta}\bar{D}\right]. \qquad (6.11)\end{aligned}$$

This momentum may be expressed as an operator acting on the mean fluid velocity,

$$\boxed{\bar{\mathbf{m}} = \left[\left(\bar{D} - \tfrac{1}{2}\,\hat{\Delta}\bar{D}\right) - \tfrac{1}{2}\left(\hat{\Delta}\bar{D}\cdot + \bar{D}\hat{\Delta}\cdot\,\right)\right]\bar{\mathbf{u}} \equiv \hat{\mathcal{O}}\bar{\mathbf{u}}\,, \qquad (6.12)}$$

which defines the operator $\hat{\mathcal{O}}$. (The first parenthesis in the square brackets contains a multiplier and the second one contains a symmetric operator.)

To make a connection between the barotropic $\overline{g\ell m}$ motion equation (6.10) and the original GLM motion equation (2.4), we shall define the mean momentum as $\bar{\mathbf{m}} = \bar{D}(\bar{\mathbf{u}} - \bar{\mathbf{p}})$ with pseudomomentum density

$$\bar{D}\bar{\mathbf{p}} \equiv \tfrac{1}{2}\left[\hat{\Delta}(\bar{D}\bar{\mathbf{u}}) + \bar{D}\hat{\Delta}\bar{\mathbf{u}} + \bar{\mathbf{u}}\,\hat{\Delta}\bar{D}\right].$$

This definition of pseudomomentum and the continuity equation for $\bar{D}$ allows equation (6.10) to be rewritten as, cf. equation (2.4),

$$\boxed{(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\big(\bar{\mathbf{u}} - \bar{\mathbf{p}}\big) + \big(\bar{u}_k - \bar{p}_k\big)\nabla\bar{u}^k - \nabla\,\bar{\Pi}^{g\ell m} + \tfrac{1}{\bar{D}}\Gamma_{kl}\nabla\overline{\xi^k\xi^l} = 0\,.}$$

The closed barotropic $\overline{g\ell m}$ system consists of the EP motion equation (6.10) and two auxiliary equations. These are the continuity equation (4.5) for $\bar{D}$ and the advection equation (4) for $\overline{\xi^k\xi^l}$, recalled as

$$\partial_t\,\bar{D} + \mathrm{div}\bar{D}\bar{\mathbf{u}} = 0 \quad \text{and} \quad (\partial_t + \bar{\mathbf{u}}\cdot\nabla)\,\overline{\xi^k\xi^l} = 0\,. \qquad (6.13)$$

The dynamical properties of the closed barotropic $\overline{g\ell m}$ system may be investigated using the EP framework. For these equations, we have the Kelvin-Noether circulation theorem, as well as conservation laws for potential vorticity, momentum and energy.

Kelvin-Noether circulation theorem for barotropic $\overline{g\ell m}$

In the EP framework, the Kelvin-Noether theorem implies the circulation relation, cf. equation (3.2) for adiabatic GLM,

$$\frac{d}{dt}\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\bar{\mathbf{m}}\cdot d\mathbf{x} = -\oint_{c(\bar{\mathbf{u}})}\frac{1}{\bar{D}}\Gamma_{k\,l}\,d\overline{\xi^k\xi^l}\,.$$

Thus, the advected Lagrangian statistical moments $\overline{\xi^k\xi^l}$ play the same role that specific entropy and relative buoyancy played in the adiabatic and stratified GLM cases treated earlier. From Stokes theorem and scalar advection of $\overline{\xi^k\xi^l}$, we also find **local potential vorticity conservation**

$$\big(\partial_t + \bar{\mathbf{u}}\cdot\nabla\big)q^{k\,l} = 0\,,\quad q^{k\,l} = \frac{1}{\bar{D}}\nabla\big(\,\overline{\xi^k\xi^l}\,\big)\cdot\mathrm{curl}\Big(\frac{1}{\bar{D}}\bar{\mathbf{m}}\Big)\,,\quad \forall\, k, l\,.$$

Momentum conservation for barotropic $\overline{g\ell m}$

Because the Lagrangian $\bar{\ell}$ in equation (6.6) is invariant under translations, Noether's theorem yields the momentum conservation law (3.5),

$$\partial_t\,\bar{m}_i + \partial_j\,\bar{T}^j_i = 0\,,$$

where $\bar{\mathbf{m}} = \delta\bar{\ell}/\delta\bar{\mathbf{u}}$ is the $\overline{g\ell m}$ Eulerian-mean momentum density in equation (6.12) and the Eulerian-mean stress tensor $\bar{T}^j_i$ is written in equation (3.6) in the form

$$\bar{T}^j_i \;=\; \bar{m}_i\bar{u}^{\,j} + \delta^j_i\Big(\bar{\mathcal{L}} - \tilde{D}\frac{\partial\bar{\mathcal{L}}}{\partial\tilde{D}}\Big)\,.$$

For the $\overline{g\ell m}$ theory, this **stress tensor** is given in terms of mean fluid quantities by

$$\boxed{\bar{T}^j_i \;=\; \bar{m}_i\bar{u}^{\,j} + \delta^j_i\,\mathcal{P} - \bar{D}_{,i}\,\overline{\xi^j\xi^k}\,\big(\,|\bar{\mathbf{u}}|^2 - h(\bar{D})\big)_{,\,k} - \bar{u}^{\,m}_{,\,i}\,\overline{\xi^j\xi^k}\,(\bar{D}\bar{u}_{\,m})_{,\,k}\,.}$$

Here $\mathcal{P}$ denotes the total $\overline{g\ell m}$ **mean pressure**,

$$\mathcal{P} = (1-\hat{\Delta})p(\bar{D}) + \tfrac{1}{2}\,\bar{D}\hat{\Delta}|\bar{\mathbf{u}}|^2 + \tfrac{1}{2}\,\overline{\xi^k\xi^l}\,\bar{D}_{,\,l}\big(\,|\bar{\mathbf{u}}|^2 + c^2(\bar{D}) + h(\bar{D})\big)_{,\,k}\,.$$

For an ideal $\gamma-$law gas, $c^2 = \gamma\,p(\bar{D})/\bar{D}$ and $c^2 + h = \gamma c^2/(\gamma - 1)$.

Energy conservation for barotropic $\overline{g\ell m}$

The Legendre transformation of the mean Lagrangian (6.6) yields the conserved mean energy, also given by Noether's theorem, as

$$\boxed{\begin{aligned}\bar{\mathcal{H}} &= \int \frac{\delta\bar\ell}{\delta\bar{\mathbf{u}}}\cdot\bar{\mathbf{u}}\,d^3x - \bar\ell(\bar{\mathbf{u}},\bar D,\overline{\xi^k\xi^l})\\ &= \int\Big[\tfrac12\,\bar D|\bar{\mathbf{u}}|^2 + \bar D e(\bar D) + \tfrac12\bar D\,\overline{\xi^k\xi^l}\,\bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + \frac{c^2(\bar D)}{2\bar D}\,\overline{\xi^k\xi^l}\,\bar D_{,k}\bar D_{,l}\Big]d^3x\end{aligned}}$$

We note that we may write the latter two terms in the mean conserved energy $\bar{\mathcal{H}}$, i.e., those due only to fluctuations, as

$$\tfrac12\overline{\mathcal{H}''} = \int\Big[\tfrac12\,\bar D\overline{|\mathbf{u}'|^2} + \frac{c^2(\bar D)}{2\bar D}\,\overline{D'^2}\Big]d^3x \neq \tfrac12\overline{E''}\,.$$

This expression does not recover the result of Blokhintsev [1945] mentioned earlier in equation (5.4). However, it has the advantage of being a positive-definite mean fluctuational energy for the closed $\overline{g\ell m}$ system.

Lie-Poisson Hamiltonian formulation of barotropic $\overline{g\ell m}$

Being an EP system, the barotropic $\overline{g\ell m}$ theory may be transformed into Lie-Poisson Hamiltonian form, by following the procedure explained in Holm, Marsden & Ratiu [1998a]. This Hamiltonian formulation begins by writing the Legendre-transformed energy in terms of the momentum. We shall assume the operator $\hat{\mathcal{O}}$ in momentum-velocity relation (6.12) is invertible, so that one may solve for the velocity from the momentum as $\bar{\mathbf{u}} = \hat{\mathcal{O}}^{-1}\bar{\mathbf{m}}$. Thus, the energy Hamiltonian for

$$\bar{\mathcal{H}} \;=\; \int\Big[\tfrac12\,\bar{\mathbf{m}}\cdot\hat{\mathcal{O}}^{-1}\bar{\mathbf{m}} + \bar D e(\bar D) + \tfrac{c^2(\bar D)}{2\bar D}\,\overline{\xi^k\xi^l}\,\bar D_{,k}\bar D_{,l}\Big]d^3x\,.$$

The ideal barotropic $\overline{g\ell m}$ equations may now be treated in the Lie-Poisson Hamiltonian framework, if so desired. The corresponding Lie-Poisson bracket is of the standard type, defined on the dual of a certain semidirect-product Lie algebra, as described, e.g., in Holm, Marsden, Ratiu & Weinstein [1985]. See also Marsden & Ratiu [1999] for an introduction to this now-standard theory and references to the literature.

Barotropic $\overline{g\ell m}-\alpha$, a simplification of barotropic $\overline{g\ell m}$ for constant isotropic Lagrangian statistics

The scalar advection equation (6.13) for the Lagrangian statistical moments admits the constant isotropic solution $\overline{\xi^k\xi^l} = \alpha^2\,\delta^{kl}$ where α is a constant length scale. The $\overline{g\ell m}$ Lagrangian (6.6) in this case simplifies to, cf. the Lagrangian (6.2) for the incompressible Euler-alpha model,

$$\boxed{\begin{aligned}\bar\ell \;&=\; \int \Big[\tfrac{1}{2}\bar D|\bar{\mathbf{u}}|^2 - \bar D e(\bar D)\Big)\Big]\,d^3x \qquad (6.14)\\ &\quad + \alpha^2\!\int \Big[\tfrac{1}{2}\bar D\,|\nabla\mathbf{u}|^2 + \tfrac{1}{2}\nabla|\bar{\mathbf{u}}|^2\;\cdot\nabla\bar D - \tfrac{c^2(\bar D)}{2\bar D}\,|\nabla\bar D|^2\Big]\,d^3x\,.\end{aligned}}$$

This is the Lagrangian for the compressible $\overline{g\ell m}-\alpha$ model with constant length scale α. For constant α, the generalized Laplacian $\hat\Delta$ in the previous equations reduces to $\hat\Delta\to\alpha^2\Delta$, where Δ is the ordinary Laplacian. The result is a **compressible generalization of the Euler-alpha model**. The equations of motion for this model are (6.10), (6.12) and (6.13) with $\overline{\xi^k\xi^l}=\alpha^2\delta^{kl}$ and $\hat\Delta\to\alpha^2\Delta$ for constant α.

Barotropic $g\ell m$ models in a rotating frame

The EP setting is convenient for transforming the averaged Lagrangian $\bar\ell$ into a rotating frame. One defines the rotation vector potential $\mathbf{R}$ satisfying $\operatorname{curl}\mathbf{R} = 2\Omega(\mathbf{x})$, for a spatially dependent rotation frequency $\Omega(\mathbf{x})$. The transformation begins by substituting into the original Lagrangian the linearized relation $\bar{\mathbf{u}}+\mathbf{u}' = \bar{\mathbf{u}}^*+\mathbf{u}'^*+\bar{\mathbf{R}}+\mathbf{R}'$, with $\mathbf{R}'=\mathbf{R}^\ell-\xi\cdot\nabla\bar{\mathbf{R}}$. One then averages and finally drops the asterisk in $(\,\cdot\,)^*$ to find, cf. equation (6.4),

$$\begin{aligned}\bar\ell \;&=\; \int \Big[\tfrac{1}{2}\bar D|\bar{\mathbf{u}}+\bar{\mathbf{R}}|^2 - \bar D e(\bar D)\Big)\Big]\,d^3x \qquad (6.15)\\ &+\!\int\Big[\tfrac{1}{2}\bar D\overline{|\mathbf{u}'|^2}+\tfrac{1}{2}\bar D\overline{|\mathbf{R}'|^2}+\overline{D'(\mathbf{u}'+\mathbf{R}')}\cdot\big(\bar{\mathbf{u}}+\bar{\mathbf{R}}\big)-\frac{1}{2}\frac{c^2(\bar D)}{\bar D}\overline{D'^2}\Big]\,d^3x\,.\end{aligned}$$

For a *constant* rotation frequency, $\bar{\mathbf{R}}=\Omega\times\mathbf{x}$ and $\mathbf{R}'$ vanishes. In this Lagrangian, the velocities $\bar{\mathbf{u}}$ and $\mathbf{u}'$ are measured in the rotating frame. The analysis then proceeds as in the earlier sections in the EP setting. See, e.g., equations (3.2), (3.7), (5.2), (6.6) and (6.7).

Outlook.

- We shall discuss the effects of rotation and its interaction with fluctuations elsewhere, and compare with Bühler and McIntyre [1998] in the Boussinesq stratified flow context. In the simpler shallow water context, one may use the Green's function method of section 4 to test the closure hypotheses in equation (6.5). This will be reported in Holm, Tartakovsky, Wingate and Winter [2001].

- The analysis of the one-dimensional barotropic $\overline{g\ell m}-\alpha$ equations (without rotation) will also be studied elsewhere, in Holm, Lowrie and Wingate [2001].

- The proper choice of dissipation for this system deserves further investigation. We propose to add dissipation as **shear viscosity** in the form
$$\partial_t \,\bar{m}_i + \partial_j \,\bar{T}_i^j = \frac{\partial}{\partial x^j}\Big(\nu(\bar{D} - \mathcal{O})u_{i\,,j}\Big)\,,$$
where $\mathcal{O}$ is the positive symmetric operator in equation (6.12). This choice assures monotonic decay of the energy in (6.14).

7 Conclusions

Summary remarks.

- **Remarkable opportunity.** The EP Averaging Lemma implies that the GLM-averaged Euler equations appear when the EP variational principle is applied to the GLM-averaged Lagrangian. This is part of a general theorem discussed in Holm [2001].

- **Theme.** We applied small-disturbance ideas that are also basic for fluid stability analysis and rapid distortion theory in a variational approach that uses Lagrangian displacements and GLM averaging in the Euler-Poincaré (EP) framework.

- **Traditional stability analysis.** The Lagrangian small-disturbance approach is fundamental and has a tradition in stability analysis – although the work here departs from that tradition, because we also average the second-variation Lagrangian and we vary with respect to different quantities than in stability analysis.

- **Linearized fluctuation relations.** The approach used here linearizes the expressions that relate the fluctuations of a given fluid quantity around its Eulerian mean and the fluctuating displacement of a Lagrangian fluid parcel trajectory around its mean position. (These standard linearized fluctuation equations are expressed geometrically in terms of Lie derivatives in Appendix #2 in section 9.)

  Use of the linearized fluctuation relations brings gradients of Eulerian mean fluid quantities into the second-variation Lagrangian for the EP variational principle that leads to the $g\ell m$ equations. We emphasized that the linear $\mathbf{u}'-$fluctuation relation provides a useful ingredient for modeling the effects of fluctuations on the mean. The effects of these fluctuations in the resulting EP equations, however, have both linear and nonlinear aspects. (We note that their linear aspects may also be *nonlocal*.)

- **Lagrangian statistics.** GLM and its small-amplitude approximation $g\ell m$ both shift the closure problem from Eulerian velocity correlations to the statistics of Lagrangian displacements. However, these Lagrangian statistics are not readily available or easily observable. We explored how these statistics might be inferred by solving the linearized fluctuation equations using a Green's function method. This Green's function method also suggests some opportunities for diagnosing the Lagrangian displacement statistics in a DNS, or, perhaps, in data assimilation.

  Experiments for measuring Lagrangian statistics are under development. See, e.g., E. Bodenschatz et al. [2000].

- **Green's function diagnostic.** The homogeneous and inhomogeneous parts of the solution in the Green's function method correspond to frozen-in, and propagating fluctuations.

  We proposed an explicit method for using the Green's function associated with the linearized fluctuation equations to diagnose the propagating Lagrangian fluctuations in a direct numerical simulation by using measurements of the Eulerian fluctuations, $\mathbf{u}'$ and D', as input.

- **Variational closure ideas.** The GLM and $g\ell m$ theories have the same advantages of the EP mathematical structure and also the

same disadvantages of not being closed. Three ideas for closure assumptions are suggested by the second-order $g\ell m$ equations:

(1) Not unexpectedly, these closure approximations depend on gradients of Eulerian mean fluid quantities.

(2) The $\mathbf{u}'-$ velocity fluctuation relation plays a central role in modeling the necessary **Taylor hypotheses**.

(3) The closure approximations may be introduced into the EP variational principle for the $g\ell m$ equations, before its variations are taken.

- **Alpha models.** Alpha models are seen as closure models of these $g\ell m$ equations, obtained by ansatzes involving terms from the linear $\mathbf{u}'-$ and $D'-$ fluctuation equations. These ansatzes are introduced into the EP variational principle for the $g\ell m$ equations, before its variations are taken. Three of these models have simple "frozen-in" evolution equations for determining $\overline{\xi\xi}$. The correlation $\overline{\xi\xi}$, in turn, appears in the nonlinearity of the motion equation for these alpha models.

- **Nonlinear feedback.** The linear approximation of the fluctuation equations leads to a nonlinear feedback in the closed motion equations that may tend to keep them within their range of validity. In particular, the mean fluctuation energy contains gradients of the mean fluid quantities. Because of this energy penalty, large mean gradients will tend not to develop in the course of the mean dynamics.

- **Bridge.** The alpha model and some of its variants can also be a motivation for further developing the $g\ell m$ equations. There are several variants of alpha-models arising from ansatzes in a Taylor expansion approximation of terms in an averaged Lagrangian. One could perhaps use the Green's function method and $g\ell m$ theory to test some of these hypotheses, since they tend to have the same ingredients as in the $\mathbf{u}'-$equation. In this sense, the $g\ell m$ theory may be available as a "bridge" between the alpha models and the exact nonlinear GLM theory.

  Of course, much remains to be done in this regard. However, the $g\ell m$ framework seems to offer a promising new opportunity for

modeling the nonlinearity of fluid turbulence from a Lagrangian perspective.

- **Synopsis.** This paper connects the GLM equations to the Euler alpha models through a new set of Eulerian mean $g\ell m$ fluid equations that are derived in the small amplitude limit of the GLM equations. These equations comprise a one-point closure approximation that is second-order in the Lagrangian fluctuation statistics. In principle, these equations may be closed by using the linearized dynamics of the original equations in combination with the linearized fluctuation relations. However, because of the complexity still remaining even at the intermediate $g\ell m$ level, we sought simpler closures by using these linearized fluctuation relations to guide our choices among the various **Taylor hypotheses** for deriving variants of the Euler-alpha models and related models in a variational closure procedure. Following the original procedure discussed in Holm, Marsden & Ratiu [1998a] for deriving the Euler-alpha model, one substitutes these linear versions of the Taylor hypothesis into the Lagrangian before taking its Eulerian mean and then its variations in the EP framework. This procedure preserved the Kelvin-Noether circulation theorem, which we regard as a basic geometrical property of all ideal fluid models. This procedure also preserved the mean momentum and energy balances. Finally, the procedure led to a barotropic compressible generalization of the Euler-alpha models.

7.1 Summary

In this paper, we developed the geometric approach to dimension reduction especially for models of turbulence in weakly compressible fluids in the context of GLM averaging. Our approach concentrated on reduction of the Lagrangian in Hamilton's principle for adiabatic compressible fluid dynamics by using a combination of compatible symmetries and averaging in the EP framework. This approach is versatile enough to include ocean circulation models for global climate modeling, as well as fundamental research in turbulence. The present paper analyzes the basic equations in the framework of the EP theory and thereby presents them in a unified geometrical context for further application.

The EP Averaging Lemma establishes the equivalence of modeling

using the GLM approach, either by directly averaging the equations of motion, or by averaging the Lagrangian for these equations before taking its variations.

We discussed EP formulations of both Lagrangian-mean, and Eulerian-mean fluid equations for modeling turbulence.
We used various elements of the classical theory of turbulence, including:

- Reynolds decomposition(s),
- Taylor hypothesis closures (THC),
- Hamilton's principle,
- Averaged Lagrangians and
- Euler-Poincaré equations

to model and analyze the mean dynamical effects of fluctuations on 3D exact Lagrangian-mean and approximate second-order Eulerian-mean fluid motion.

Our starting point was the exact nonlinear Generalized Lagrangian Mean (GLM) equations of Andrews & McIntyre [1978a] for a compressible adiabatic fluid. We first recast the GLM equations as EP equations resulting from the Lagrangian mean of Hamilton's principle, written in the Eulerian fluid description. This demonstrated the validity of the general principles underlying the EP Averaging Lemma. We then used the small-amplitude approximation to linearize the relations between Lagrangian disturbances and Eulerian fluctuations. We substituted these linearizations into Hamilton's principle for the GLM equations and kept terms up to quadratic order before taking the Eulerian mean. The EP equations resulting from this approximate Eulerian-mean Lagrangian produced a new set of $g\ell m$ equations. These comprise a second-order (one-point, weakly compressible) turbulence closure model that captures some aspects of the influence of the small scale dynamics on the large scale flow – while preserving the mathematical structure of the original Euler equations.

We observed that $g\ell m$ theory relates certain combined Eulerian and Lagrangian aspects of wave properties through expressions also involving gradients of mean flow properties. This observation suggested we consider closure schemes that involve substituting approximations or truncated versions of these relations between wave properties and mean gradients into Hamilton's principle, before taking its variations. Thus, we regarded these approximated, or truncated, relations as a type

of **Taylor hypothesis closure**. We tried one of the simpler variants of this idea and found a new compressible generalization of the Euler-alpha models, the $\overline{g\ell m}$ closure, whose solution behavior remains to be studied.

Thus, introducing such Taylor-hypotheses into the second-order Eulerian-mean closure approximation for the $g\ell m$ theory led to variants of the Euler-alpha models, and a framework for exploring other options. This included finding new variants of them for compressible flows that we discussed in section 6.4.

Being derivable in the EP framework, the GLM theory, as well as its second-order Eulerian-mean closure approximation, the new $g\ell m$ theory, and the new compressible $\overline{g\ell m}$ generalization of the Euler-alpha equations, all possess the same fundamental structure and underlying geometry that are shared by all other ideal fluid theories in the EP framework. This geometrical structure ensures that these fluid theories (both exact and approximate ones) each retains its own Kelvin circulation theorem and the associated conservation law for potential vorticity arising from it by Noether's theorem, for particle relabeling symmetry. The EP framework also implies balance laws for momentum and energy exchanges between mean flow and wave properties.

The geometrical structure of the EP framework leads, in addition, to the Lie-Poisson Hamiltonian formulation for GLM theory, its Eulerian-mean closure approximation and the variants of the Euler-alpha models. This Hamiltonian formulation possesses potential-vorticity Casimirs associated with its Lie-Poisson bracket. In turn, the Lie-Poisson Hamiltonian structure leads to the energy-Casimir method for characterizing equilibrium solutions as critical points of a constrained energy and for establishing their nonlinear Liapunov stability conditions. All of these additional features are now available,for the GLM theory, for its Eulerian-mean closure approximation, the $g\ell m$ theory, for the compressible $\overline{g\ell m}$ closure model, and also for the alpha models and any new variants of them that may arise in the future.

The $g\ell m$ theory provides a bridge that spans from the alpha models to the exact nonlinear GLM theory. We hope this bridge will be useful in answering questions that arise in the context of the alpha models and other turbulence closure models.

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8 Appendix #1: The Euler-Poincaré theorems for fluids with advected properties

8.1 Mathematical setting and statement of the EP theorem

The assumptions of the Euler-Poincaré theorem from Holm, Marsden & Ratiu [1998a] are briefly listed below.

- There is a *right* representation of Lie group G on the vector space V and G acts in the natural way on the *right* on $TG \times V^*$: $(v_g, a)h = (v_g h, ah)$.
- Assume that the function $L : TG \times V^* \to \mathbf{R}$ is right G–invariant.
- In particular, if $a_0 \in V^*$, define the Lagrangian $L_{a_0} : TG \to \mathbf{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of the right action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.
- Right G–invariance of L permits one to define $\ell : \mathbf{g} \times V^* \to \mathbf{R}$ by

  $$\ell(v_g g^{-1}, a_0 g^{-1}) = L(v_g, a_0).$$

  Conversely, this relation defines for any $\ell : \mathbf{g} \times V^* \to \mathbf{R}$ a right G–invariant function $L : TG \times V^* \to \mathbf{R}$.
- For a curve $g(t) \in G$, let

  $$u(t) \equiv \dot{g}(t) g(t)^{-1} \in TG/G \cong \mathbf{g}$$

  and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients

  $$\dot{a}(t) = -a(t)u(t) \tag{8.1}$$

  where the action of $u \in \mathbf{g}$ on the initial condition $a(0) = a_0 \in V^*$ is denoted by concatenation from the right. The solution of (8.1) can be written as the **advective transport relation**,

  $$a(t) = a_0 g(t)^{-1} \,.$$

Theorem 8.1 (EP Theorem) *The following are equivalent:*

i *Hamilton's variational principle*

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0 \tag{8.2}$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

ii *$g(t)$ satisfies the Euler–Lagrange equations for L_{a_0} on G.*

iii *The constrained variational principle*

$$\delta \int_{t_1}^{t_2} \ell\,(u(t), a(t))dt = 0 \tag{8.3}$$

holds on $\mathbf{g} \times V^*$, using variations of the form

$$\delta u = \frac{\partial \eta}{\partial t} + \mathrm{ad}_u\, \eta, \quad \delta a = -a\, \eta, \tag{8.4}$$

where $\eta(t) \in \mathbf{g}$ vanishes at the endpoints.

iv *The Euler–Poincaré equations hold on $\mathbf{g} \times V^*$*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} = -\mathrm{ad}^*_u \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta a} \diamond a. \tag{8.5}$$

Remarks.

- The EP motion equation and advection relations may also be written equivalently using Lie derivative notation as

$$\Big(\frac{\partial}{\partial t} + \pounds_u\Big)\frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta a} \diamond a = 0\,, \quad \text{and} \quad \Big(\frac{\partial}{\partial t} + \pounds_u\Big)a = 0\,.$$

  The equivalence here of $\pounds_u$ and ad^*_u arises because $\delta\ell/\delta u$ is a one-form density and the equality $\mathrm{ad}^*_u \mu = \pounds_u \mu$ holds for any one-form density μ.

- In the Lie derivative notation, one proves the **Kelvin-Noether circulation theorem** immediately as a corollary, by

$$\frac{d}{dt}\oint_{c(u)} \frac{1}{D}\,\frac{\delta \ell}{\delta u} = \oint_{c(u)} \Big(\frac{\partial}{\partial t} + \pounds_u\Big)\frac{1}{D}\,\frac{\delta \ell}{\delta u} = \oint_{c(u)} \frac{1}{D}\,\frac{\delta \ell}{\delta a} \diamond a\,,$$

  for any closed curve $c(u)$ that moves with the fluid and advected density D.

8.2 Summary of the EP equations in geometric notation for a barotropic ideal fluid

As proven in Holm, Marsden & Ratiu [1998a], the Euler-Lagrange equations of motion on $(TG \times V^*)$ for a Lagrangian that is invariant under the group G (where G is the diffeomorphism group for fluids) are equivalent to the **EP equations**, expressed for a barotropic ideal fluid as

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u^i} + \underbrace{\frac{\partial}{\partial x^j}\Big(\frac{\delta \ell}{\delta u^i}u^j\Big) + \frac{\delta \ell}{\delta u^j}\frac{\partial u^j}{\partial x^i}}_{\equiv\left(\mathrm{ad}^*_u\frac{\delta \ell}{\delta u}\right)_i} - \underbrace{D\frac{\partial}{\partial x^i}\Big(\frac{\delta \ell}{\delta D}\Big)}_{\equiv\left(\frac{\delta \ell}{\delta D}\diamond D\right)_i} = 0\,.$$

Here the density $D \in V^*$ satisfies the advection relation,

$$\frac{\partial D}{\partial t} = -\,\mathrm{div}(D\mathbf{u})\,.$$

In **EP geometric notation** the last two equations are written equivalently as

$$\Big(\frac{\partial}{\partial t} + \mathrm{ad}^*_u\Big)\frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta D}\diamond D = 0\,, \quad \text{and} \quad \frac{\partial D}{\partial t} = -\,\pounds_u D\,,$$

where $\pounds_u$ denotes the Lie derivative with respect to velocity u, and the operations ad^* and $\diamond$ are defined using the L_2 pairing $\langle f, g\rangle = \int fg\,d^3x$, as

$$-\Big\langle \mathrm{ad}^*_u\frac{\delta \ell}{\delta u}, \xi\Big\rangle = \Big\langle \frac{\delta \ell}{\delta u}, \mathrm{ad}_u\xi\Big\rangle$$

$$\mathrm{ad}_u\xi = \xi u - u\xi = u\cdot\nabla\xi - \xi\cdot\nabla u = -\,\mathrm{ad}_\xi u$$

$$-\Big\langle \frac{\delta \ell}{\delta D}\diamond D, \xi\Big\rangle = \Big\langle \frac{\delta \ell}{\delta D}, \pounds_\xi D\Big\rangle = \Big\langle \frac{\delta \ell}{\delta D}, \mathrm{div}(D\xi)\Big\rangle\,.$$

Remark.

- The EP equation may also be written using the Lie derivative as

$$\Big(\frac{\partial}{\partial t} + \pounds_u\Big)\frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta D}\diamond D = 0\,, \quad \text{and} \quad \Big(\frac{\partial}{\partial t} + \pounds_u\Big)D = 0\,.$$

  This equivalence using either $\pounds_u$ or ad^*_u arises because $\delta\ell/\delta u$ is a one-form density and the equality $\mathrm{ad}^*_u\mu = \pounds_u\mu$ holds for any one-form density μ.

See Appendix #2 (section 9) for more discussion of how this underlying geometry expresses itself in the linearized $g\ell m$ approximation.

9 Appendix #2: Geometric interpretations of the linearized fluctuation formulas

In fluid dynamics the state space is the group of diffeomorphisms – the smooth invertible maps that take the reference configuration of the fluid with coordinates $\mathbf{x}_0$ into its current configuration $\mathbf{x}(t, \mathbf{x}_0)$ in the container.

9.1 Lagrangian & Eulerian pictures

Lagrangian fluid trajectories are *orbits* under the action of the diffeomorphism group $G = Diff$ parameterized by time t, thus,

$$\boldsymbol{x}(t, x_0) = g(t) \cdot x_0\,, \quad x_0 = g^{-1}(t) \cdot \boldsymbol{x}(t)\,, \quad g \in Diff\,.$$

The corresponding *velocity relations* are

$$\dot{\boldsymbol{x}}(t, x_0) = \dot{g}(t) \cdot x_0 = \dot{g}\, g^{-1}(t) \cdot \boldsymbol{x} = \boldsymbol{u}(\boldsymbol{x}, t)\,.$$

This formula relates the Lagrangian and Eulerian definitions of velocity.

For *variations*, one introduces another parameter ϵ, and denotes,

$$g(t, \epsilon)\ : \quad \frac{\partial g}{\partial t} = \dot{g}(t, \epsilon)\,, \quad \frac{\partial g}{\partial \epsilon} = g'(t, \epsilon)\,.$$

Thus, the corresponding *variational relations* are

$$\boldsymbol{x}'(t, \epsilon, x_0) = g'(t, \epsilon) \cdot x_0 = g'\, g^{-1}(t, \epsilon) \cdot \boldsymbol{x} = \boldsymbol{\xi}(\boldsymbol{x}, t, \epsilon)\,.$$

This formula relates the Lagrangian and Eulerian definitions of spatial trajectory fluctuations.

Thus, at spatial position $\boldsymbol{x}$ and time t, a given Lagrangian trajectory has **two tangent vectors**: the Eulerian velocity, $\boldsymbol{u} = \dot{g}\, g^{-1}$, and the Eulerian fluctuation/variation, $\boldsymbol{\xi}(\boldsymbol{x}, t) = g'\, g^{-1}$.

These Lagrangian considerations lead to the following geometric interpretations of the linearized relations for spatial trajectory fluctuations in terms of the displacement fluctuation, ξ.

9.2 The $\boldsymbol{u}'-$ equation

Equality of cross derivatives in the difference

$$(\dot{g}\, g^{-1})' - (g'\, g^{-1})^{\bullet}$$

gives

$$\boldsymbol{u}'(\boldsymbol{x},t) - \frac{\partial \xi}{\partial t} = \xi u - u\xi = u \cdot \nabla \xi - \xi \cdot \nabla u = -\,\mathrm{ad}_\xi u$$

This is the $\boldsymbol{u}'-$ equation,

$$\boldsymbol{u}'(\boldsymbol{x},t) = \frac{\partial \xi}{\partial t} + u \cdot \nabla \xi - \xi \cdot \nabla u = \partial_t\, \xi - \;\mathrm{ad}_\xi u\,,$$

which relates Eulerian velocity variations $\boldsymbol{u}'(\boldsymbol{x},t)$ and Lagrangian trajectory variation tendencies $\partial_t\, \xi(\boldsymbol{x},t)$. This is a key equation for making the $g\ell m$ approximations in the GLM Lagrangian. The particular solutions of the $\boldsymbol{u}'-$ equation also play a role as sources of inspiration for Taylor hypotheses in simplifying the $g\ell m$ equations to obtain the alpha models in section 6.

For a discussion of the linearized fluctuation relations from the viewpoint of Lagrangian stability analysis with a similar geometric viewpoint, see Friedman & Schutz [1978a, 1978b].

9.3 Advected quantities

For **advected quantities** the right action of the group, denoted as

$$a(t,\epsilon) = a_0 g^{-1}(t,\epsilon)\,,$$

implies the *Eulerian advection* ($^{\bullet}$ at fixed $\boldsymbol{x}$)

$$\dot{a}(\boldsymbol{x}) = -\, a_0 g^{-1} \dot{g} g^{-1} = -a\, u = -\pounds_u\, a\,,$$

and the *Eulerian variation* ($'$ at fixed $\boldsymbol{x}$)

$$\delta a(\boldsymbol{x}) = a'(\boldsymbol{x}) = -\, a_0 g^{-1} g' g^{-1} = -a\, \xi = -\pounds_\xi\, a\,.$$

> **Remark.** This is the general result, examples of which were given earlier. For an advected scalar χ one finds $\chi' = -\pounds_\xi\, \chi = -\,\xi \cdot \nabla \chi$ and for an advected density $\bar{D}$ one

finds $D' = -\pounds_\xi \bar{D} = -\operatorname{div} \bar{D}\xi$. We note that – from their definitions in terms of Taylor series approximations – all of these linearized fluctuation relations introduce gradients of Eulerian mean fluid quantities.

Thus, the smooth invertible $\epsilon-$dependence representing variations in the $g\ell m$ and GLM theories is generated by the displacement vector field ξ, just as the time dependence is generated by the fluid velocity.

10 Appendix #3: On WMFI Decomposition (Lagrangian mean plus fluctuations)

The WKB wave packet form of ξ is given by (assuming $\epsilon \ll 1$, $\alpha \ll 1$)

$$\mathbf{x}^\xi = \mathbf{x} + \alpha\,\xi(\mathbf{x},t) = \mathbf{x} + \alpha\left(\mathbf{a}e^{i\phi/\epsilon} + cc\right), \quad \mathbf{a} = \mathbf{a}(\epsilon\mathbf{x}, \epsilon t)\,, \quad \phi = \phi(\epsilon\mathbf{x}, \epsilon t)$$

The **WMFI Expansion of the Fluid Lagrangian** yields terms at several orders of α and ϵ,

$$\mathcal{L}^{(\alpha,\epsilon)} = \underbrace{\mathcal{L}^{(0,0)}}_{mean flow} + \underbrace{\alpha^2\Big[\mathcal{L}^{(2,0)} + \epsilon\mathcal{L}^{(2,1)} + \epsilon^2\mathcal{L}^{(2,2)}\Big]}_{WMFI} + \underbrace{O(\alpha^4)}_{Wave-Wave}$$

Thus there are several levels of approximation available from this asymptotic expansion of **Hamilton's principle(s)**. We simply list these as:

(1) Set $\alpha = 0$ and vary mean flow (MF) quantities
$\Rightarrow$ Euler's equations for mean flow when $\delta\mathcal{L}^{(0,0)} = 0$,
as Euler-Lagrange equations in EP form

(2) Vary only *wave* quantities, evaluate MF at $\delta\mathcal{L}^{(0,0)} = 0$
$\Rightarrow$ Linearized spectral stability equations,
cf. Andrews & McIntyre [1978b]

(3) Phase average, then vary only wave quantities, and evaluate MF at $\delta\mathcal{L}^{(0,0)} = 0$
$\Rightarrow$ Whitham's modulation equations, at order $O(\alpha^2, \alpha^2\epsilon)$
$\Rightarrow$ WKB stability equations, at order $O(\alpha^2\epsilon^2)$, see Lifshitz [1994] and Bayly et al. [1996].

(4) Phase average at constant Lagrangian coordinate, then vary only MF quantities, and *prescribe* ξ
$\Rightarrow$ GLM theory.

(5) Phase average, then vary both wave *and* MF quantities,
$\Rightarrow$ WMFI equations.

Remark. Details of the self-consistent theories of Lagrangian-mean WMFI at these various levels of approximation are given in Gjaja & Holm [1996]. In particular, the self-consistent WMFI equations are shown in Gjaja & Holm [1996] to comprise a two-fluid theory, reminiscent of Landau's theory of superfluid Helium.

11 Appendix #4: GLM, $g\ell m$, and $\alpha-$models of ideal MHD

GLM ideal MHD

This appendix highlights some of the GLM, $g\ell m$, and $\alpha-$models results in the rest of the paper by setting up the corresponding results for ideal MHD (MagnetoHydroDynamics).

In ideal MHD one begins by including the magnetic energy potential energy in the averaged Lagrangian, $\bar{\ell}(\bar{\mathbf{u}}^L, \tilde{D}, \bar{s}^L)$ in (3.2),

$$P_{mag} = \int \tfrac{1}{2}\overline{|\mathbf{B}^\xi|^2}\, d^3x\,.$$

The variation of this GLM-averaged magnetic potential energy gives

$$\delta P_{mag} = \int \overline{\mathbf{B}^\xi \cdot \delta\mathbf{B}^\xi}\, d^3x = \int \overline{\mathbf{B}^\xi \cdot \mathsf{K}^{-1}} \cdot \delta\tilde{\mathbf{B}}\, d^3x + \text{terms in } \delta\xi\,.$$

Here we recalled that $\tilde{\mathbf{B}} = \mathsf{K}\cdot\mathbf{B}^\xi$ is a mean quantity and used $\mathbf{B}^\xi = \mathsf{K}^{-1}\cdot\tilde{\mathbf{B}}$.

Hence, the effect of GLM averaging in MHD is to introduce a mean *tensor permeability* due to the fluctuations as

$$\tilde{\mathbf{H}} \equiv \frac{\delta P_{mag}}{\delta\tilde{\mathbf{B}}} = \overline{\mathbf{B}^\xi\cdot\mathsf{K}^{-1}} = -\,\frac{\delta\bar{\ell}}{\delta\tilde{\mathbf{B}}}\,.$$

The EP equation for ideal *barotropic* MHD is given in Holm, Marsden & Ratiu [1998a] in the present notation as

$$\frac{\partial}{\partial t}\frac{\delta\bar\ell}{\delta\bar{\mathbf{u}}^L} + \frac{\partial}{\partial x_k}\Big(\frac{\delta\bar\ell}{\delta\bar{\mathbf{u}}^L}\bar u^{L\,k}\Big) + \frac{\delta\bar\ell}{\delta\bar u^{L\,k}}\nabla\bar u^{L\,k} = \tilde D\nabla\frac{\delta\bar\ell}{\delta\tilde D} + \tilde{\mathbf{B}}\times\,\mathrm{curl}\,\frac{\delta\bar\ell}{\delta\tilde{\mathbf{B}}}\,.$$

The effect of the magnetic field is to add a $\tilde{\mathbf{J}}\times\tilde{\mathbf{B}}$ force, with

$$\tilde{\mathbf{J}} = \mathrm{curl}\,\overline{\big(\mathbf{B}^\xi\cdot\mathsf{K}^{-1}\big)}\,.$$

The auxiliary equations are the continuity equation (2.1) for $\tilde D$ and the frozen-in flux rule (2.2) for $\tilde{\mathbf{B}}$. That is,

$$\partial_t\,\tilde D + \mathrm{div}\tilde D\bar{\mathbf{u}}^L = 0\,,\quad \partial_t\,\tilde{\mathbf{B}} = \mathrm{curl}\,(\bar{\mathbf{u}}^L\times\tilde{\mathbf{B}})\quad\text{with}\quad \mathrm{div}\,\tilde{\mathbf{B}} = 0\,.$$

Remarks.

- The GLM averaged EP variational principle yields the GLM averaged equation for ideal isentropic MHD in Cartesian coordinates as

$$\boxed{\frac{D^L}{Dt}\big(\bar{\mathbf{u}}^L - \bar{\mathbf{p}}\big) + \big(\bar u^L_k - \bar p_k\big)\nabla\bar u^L_k + \nabla\,\Pi = \mathrm{curl}\,\overline{\big(\mathbf{B}^\xi\cdot\mathsf{K}^{-1}\big)}\times\tilde{\mathbf{B}}\,.}$$

  Here the **pseudomomentum vector**, $\bar{\mathbf{p}}$, is defined to be $\bar{\mathbf{p}} \equiv -\,\overline{u^\ell_k\nabla\xi^k}$, as in equation (2.5). The mean potential Π has the form of equation (2.6),

$$\Pi = \overline{e(D^\xi)} + \overline{(p^\xi/D^\xi)} + \bar\Phi^L(\mathbf{x}) - \frac{1}{2}\overline{|\mathbf{u}^\xi|^2}\,.$$

  The computation of Π uses $\tilde D = D^\xi\mathcal{J}$ from mass conservation and the Eulerian mean of the thermodynamic First Law following a fluid parcel in an isentropic compressible fluid, as in equation (2.7)

$$\boxed{d\,\overline{e(D^\xi,\bar s^L)} = -\,\frac{1}{\tilde D}\,\overline{(p^\xi\,d\mathcal{J})} + \frac{1}{\tilde D}\,\overline{(p^\xi/D^\xi)}\,d\tilde D\,.}$$

- The GLM averaging process preserves the transport structure of the original ideal MHD equations. In particular, it also yields

preserved linking numbers – **magnetic helicity and cross helicity** – involving the frozen-in averaged magnetic field and the Lagrangian mean velocity. Thus, GLM averaging preserves not the magnetic linking numbers themselves, but the *property* that the GLM averaged dynamics has preserved linking numbers.

Ideal $g\ell m$ MHD

The linearized Eulerian/Lagrangian fluctuation relation for a magnetic field is

$$\mathbf{B}' = -\,\pounds_\xi\,\bar{\mathbf{B}} = \mathrm{curl}\,(\,\xi\times\bar{\mathbf{B}}) = -\xi\cdot\nabla\bar{\mathbf{B}} + \bar{\mathbf{B}}\cdot\nabla\xi - \bar{\mathbf{B}}\,\mathrm{div}\,\xi\,.$$

Hence, the $g\ell m$ ideal MHD energy variation may be computed as

$$\delta\int \tfrac{1}{2}\overline{|\mathbf{B}'|^2}\,d^3x = -\int \delta\bar{\mathbf{B}}\cdot\overline{\Big(\xi\times\ \mathrm{curl}\,(\,\xi\times\bar{\mathbf{B}})\Big)}\,d^3x\,.$$

The motion equation for a $g\ell m$ theory of ideal MHD energy is obtained by including this variational derivative in the $\bar{\mathbf{J}}\times\bar{\mathbf{B}}$ force for the EP equation above, where,

$$\bar{\mathbf{J}} = \mathrm{curl}\,(\bar{\mathbf{B}} - \overline{(\xi\times\mathbf{B}')}\,, \quad \text{with} \quad \mathbf{B}' = \mathrm{curl}\,(\,\xi\times\bar{\mathbf{B}})\,.$$

The latter is a familiar combination from Lagrangian MHD stability analysis, see, e.g., Bernstein et al. [1958]. (This was kindly pointed out to the author by E. Caramana.)

EP $g\ell m$ equations for incompressible MHD

The total mean Lagrangian for the incompressible $g\ell m$ MHD flow is

$$\bar{\ell} = \int\Big[\tfrac{1}{2}\bar{D}\Big(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}\Big) + \bar{p}(1-\bar{D}) - \tfrac{1}{2}\Big(|\bar{\mathbf{B}}|^2 + \overline{|\mathbf{B}'|^2}\Big)\Big]\,d^3x\,. \tag{11.1}$$

The corresponding variational derivatives are given by, cf. equation (5.11),

$$\begin{aligned}\delta\bar{\ell}(\bar{\mathbf{u}},\bar{D},\bar{\mathbf{B}}) \;=\; &\int\Big[\delta\bar{D}\Big(\tfrac{1}{2}\big(\,|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2}\,\big) - \bar{p}\Big) + \delta\bar{p}\,(1-\bar{D})\\ &+\bar{D}\,\delta\bar{\mathbf{u}}\cdot\Big(\bar{\mathbf{u}} - \overline{\xi\times\mathrm{curl}\,\mathbf{u}'} + \nabla\overline{(\xi\cdot\mathbf{u}')}\Big) + \delta\bar{\mathbf{u}}\cdot\overline{\mathbf{u}'\,\mathrm{div}\,(\bar{D}\xi)}\\ &-\delta\bar{\mathbf{B}}\cdot\Big(\bar{\mathbf{B}} - \overline{\xi\times\mathrm{curl}\,\mathbf{B}'}\Big)\Big]\,d^3x\,.\end{aligned}$$

On setting $\bar{D} = 1$ and div$\xi = 0$, we obtain the same $g\ell m$ circulation velocity as in subsection 5.8,

$$\bar{\mathbf{v}} \equiv \bar{\mathbf{u}} - \overline{\xi \times \mathrm{curl}\, \mathbf{u}'} + \nabla \overline{(\xi \cdot \mathbf{u}')}\,.$$

The corresponding EP motion equation (with $\nabla \cdot \bar{\mathbf{u}} = 0$) is expressed as,

$$\frac{\partial}{\partial t} \bar{\mathbf{v}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{v}} + \bar{v}_j \nabla \bar{u}^j + \nabla \bar{\pi} = (\bar{\mathbf{J}} + \overline{\mathbf{J}''}) \times \bar{\mathbf{B}}\,, \tag{11.2}$$

with $\overline{\mathbf{J}''} = -\mathrm{curl}\, \overline{(\xi \times \mathrm{curl}\, \mathbf{B}')}$ and $\bar{\pi} = \bar{p} - \frac{1}{2}\big(|\bar{\mathbf{u}}|^2 + \overline{|\mathbf{u}'|^2} \big)$. This is the EP equation for the $g\ell m$ MHD Lagrangian (11.1). This equation also has the equivalent form,

$$\frac{\partial}{\partial t} \bar{\mathbf{v}} - \bar{\mathbf{u}} \times \mathrm{curl}\, \bar{\mathbf{v}} + \nabla \big(\bar{\mathbf{v}} \cdot \bar{\mathbf{u}} + \bar{\pi} \big) = (\bar{\mathbf{J}} + \overline{\mathbf{J}''}) \times \bar{\mathbf{B}}\,.$$

The Kelvin circulation theorem for the incompressible $g\ell m$ MHD equations is,

$$\frac{d}{dt} \oint_{c(\bar{\mathbf{u}})} \big(\bar{\mathbf{u}} - \overline{\xi \times \mathrm{curl}\, \mathbf{u}'} \big) \cdot d\mathbf{x} = \oint_{c(\bar{\mathbf{u}})} \Big((\bar{\mathbf{J}} + \overline{\mathbf{J}''}) \times \bar{\mathbf{B}} \Big) \cdot d\mathbf{x}\,. \tag{11.3}$$

Thus, for incompressible $g\ell m$ MHD the additional statistical element required for closure is

$$\overline{\mathbf{J}''} = -\,\mathrm{curl}\, \overline{\big(\xi \times \mathrm{curl}\, (\, \xi \times \bar{\mathbf{B}})\big)} = -\, \overline{\mathrm{ad}_\xi \big(\mathrm{ad}_\xi \bar{\mathbf{B}} \big)}\,, \tag{11.4}$$

where $\mathrm{ad}_\xi \bar{\mathbf{B}} = \xi \cdot \nabla \bar{\mathbf{B}} - \bar{\mathbf{B}} \cdot \nabla \xi$.

EP equations for an incompressible MHD$-\alpha$ model

Dropping derivatives of ξ again in the linearized fluctuation relations gives, cf. equation (6.5),

$$\mathbf{u}' = -\,\xi \cdot \nabla \bar{\mathbf{u}}\,, \quad D' = -\,\xi \cdot \nabla \bar{D} \quad \text{and} \quad \mathbf{B}' = -\,\xi \cdot \nabla \bar{\mathbf{B}}\,.$$

Substituting these relations into the Lagrangian (11.1) for incompressible $g\ell m$ MHD leads again to equation (11.2) with the re-definitions,

$$\bar{\mathbf{v}} = \bar{\mathbf{u}} - \hat{\Delta}\, \bar{\mathbf{u}}\,, \quad \overline{\mathbf{J}''} = -\,\mathrm{curl}\, \hat{\Delta}\, \bar{\mathbf{B}}\,, \,, \quad \hat{\Delta} = \partial_l\, \overline{\xi^k \xi^l}\, \partial_k\,.$$

The auxiliary equations are

$$\operatorname{div}\bar{\mathbf{u}} = 0\,, \quad \partial_t\,\bar{\mathbf{B}} = \operatorname{curl}\,(\bar{\mathbf{u}}\times\bar{\mathbf{B}}) \quad \text{with} \quad \operatorname{div}\bar{\mathbf{B}} = 0\,.$$

Other incompressible MHD$-\alpha$ closure options are available, corresponding to other Taylor hypotheses for $\mathbf{u}'$ and approximations for $\mathbf{B}'$. The present closure option is a straight-forward generalization of the incompressible Euler$-\alpha$ model. As one may verify, the corresponding conserved energy in this case is

$$\bar{\mathcal{E}} = \frac{1}{2}\int\Big[\ |\bar{\mathbf{u}}|^2 + \overline{\xi^k\xi^l}\ \bar{\mathbf{u}}_{,k}\cdot\bar{\mathbf{u}}_{,l} + |\bar{\mathbf{B}}|^2 + \overline{\xi^k\xi^l}\ \bar{\mathbf{B}}_{,k}\cdot\bar{\mathbf{B}}_{,l}\Big]\,d^3x\,. \tag{11.5}$$

This is the H_1 norm in both $\bar{\mathbf{u}}$ and $\bar{\mathbf{B}}$, when one chooses the subcase $\overline{\xi^k\xi^l} = \alpha^2\delta^{kl}$ and α^2 =constant. Thus, in this subcase, the presence of $\alpha \neq 0$ regularizes *both* the mean velocity and the mean magnetic field.

The incompressible $\overline{g\ell m}$ MHD case may be modified to include compressibility by following the steps for the compressible case without magnetic fields discussed in section 6.4.